Arc Tangent

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Introduction

It is shown that the arc tangent is defined by three series in its entire domain.

$$\arctan(x) = -\frac{\pi}{2} + \sum^{0 \le i < n} \frac{(-1)^{i+1}}{(2*i+1)*x^{2*i+1}}; \qquad x \le -1 \tag{1}$$

$$\arctan(x) = \sum^{0 \le i < n} (-1)^i * \frac{x^{2*i+1}}{2*i+1}; \qquad |x| \le 1 \tag{2}$$

$$\arctan(x) = \frac{\pi}{2} + \sum^{0 \le i < n} \frac{(-1)^{i+1}}{(2*i+1)*x^{2*i+1}}; \qquad x \ge 1 \tag{3}$$

The arc tangent is the unknown integral of a rational function.

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} = \frac{1}{x^2+1} \tag{4}$$

Series of Small Values

The initial steps of the division by smallest orders is determined.

$$1/(1+x^2) = 1 - x^2 + x^4 - \frac{x^6}{1+x^2} \tag{5}$$

$$\begin{array}{l} \underline{1+x^2} \\ \quad -x^2 \\ \quad \underline{-x^2 - x^4} \\ \qquad\quad x^4 \\ \qquad\quad \underline{x^4 + x^6} \\ \qquad\qquad -x^6 \end{array}$$

The division by smallest orders is determined generally.

$$\frac{1}{1+x^2} = \left(\sum^{0 \le i < n} (-1)^i * x^{2*i}\right) + \left((-1)^n * \frac{x^{2*n}}{1+x^2}\right) \tag{6}$$

A convergence test is applied.

$$\left|(-1)^i * x^{2*i}\right| > \left|(-1)^{i+1} * x^{2*i+1}\right|; \qquad |x| < 1 \tag{7}$$

The series is integrated without remainder.

$$F(x) = \sum^{0 \le i < n} (-1)^i * \frac{x^{2*i+1}}{2*i+1} \tag{8}$$

A convergence test is applied.

$$\left|(-1)^i * \frac{x^{2*i+1}}{2*i+1}\right| > \left|(-1)^{i+1} * \frac{x^{2*(i+1)+1}}{2*(i+1)+1}\right|; \qquad |x| \le 1 \tag{9}$$

The series determines the arc tangent for small values.

$$\arctan(x) = \sum^{0 \le i < n} (-1)^i * \frac{x^{2*i+1}}{2*i+1}; \qquad |x| \le 1 \tag{10}$$

Series of Large Values

The initial steps of the division by highest orders is determined.

$$1/(x^2+1) = \frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{x^6} - \frac{1}{x^6} * \frac{1}{x^2+1} \tag{11}$$

$$\begin{array}{l} \underline{1 + \frac{1}{x^2}} \\ \quad - \frac{1}{x^2} \\ \quad \underline{- \frac{1}{x^2} - \frac{1}{x^4}} \\ \qquad\qquad \frac{1}{x^4} \\ \qquad\qquad \underline{\frac{1}{x^4} + \frac{1}{x^6}} \\ \qquad\qquad\qquad - \frac{1}{x^6} \end{array}$$

The division by highest orders is determined generally.

$$\frac{1}{x^2+1} = \left(\sum^{0 < i \le n} \frac{(-1)^i}{x^{2*i}}\right) + \left(\frac{(-1)^n}{x^{2*n} * (x^2+1)}\right) \tag{12}$$

A convergence test is applied.

$$\left|\frac{(-1)^i}{x^{2*i}}\right| > \left|\frac{(-1)^{i+1}}{x^{2*(i+1)}}\right|; \qquad |x| > 1 \tag{13}$$

The series is integrated without remainder.

$$G(x) = A + \sum^{0 \le i < n} \frac{(-1)^{i+1}}{(2*i+1) * x^{2*i+1}} = A + g(x) \tag{14}$$

A convergence test is applied.

$$\left|\frac{(-1)^{i+1}}{(2*i+1)*x^{2*i+1}}\right| > \left|\frac{(-1)^{i+2}}{(2*(i+1)+1)*x^{2*(i+1)+1}}\right|; \qquad |x| \geq 1 \tag{15}$$

The integration constant A is defined by the range of the arc tangent.

$$\lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}; \qquad \lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}; \tag{16}$$

$$\lim_{x\to-\infty} g(x) = 0; \qquad \lim_{x\to\infty} g(x) = 0 \tag{17}$$

Two series of the arc tangent result that differ by the sign of the constant.

$$\arctan(x) = \pm\frac{\pi}{2} + \sum^{0\leq i<n} \frac{(-1)^{i+1}}{(2*i+1)*x^{2*i+1}}; \qquad \pm x \geq 1 \tag{18}$$

Summary

Three series determine the arc tangent in its entire domain.

$$\arctan(x) = -\frac{\pi}{2} + \sum^{0\leq i<n} \frac{(-1)^{i+1}}{(2*i+1)*x^{2*i+1}}; \qquad x \leq -1 \tag{19}$$

$$\arctan(x) = \sum^{0\leq i<n} (-1)^i * \frac{x^{2*i+1}}{2*i+1}; \qquad |x| \leq 1 \tag{20}$$

$$\arctan(x) = \frac{\pi}{2} + \sum^{0\leq i<n} \frac{(-1)^{i+1}}{(2*i+1)*x^{2*i+1}}; \qquad x \geq 1 \tag{21}$$

Both series with positive domains define $\pi/4$.

$$\arctan(1) = \sum^{0\leq i<n} \frac{(-1)^{i+1}}{2*i+1} = \frac{\pi}{4} \tag{22}$$

Short cuts exist if the value is not near the bound.

$$\arctan(x) \approx -\frac{\pi}{2} - \frac{1}{x}; \qquad x \ll -1 \tag{23}$$

$$\arctan(x) \approx x; \qquad |x| \ll 1 \tag{24}$$

$$\arctan(x) \approx \frac{\pi}{2} - \frac{1}{x}; \qquad x \gg 1 \tag{25}$$

See [2] for more details.

References

[1] Arc Tangent, Hans-Dieter Reuter, http://www.joinedpolynomials.org/arctangent.pdf

[2] Joined Polynomials, Hans-Dieter Reuter, http://www.joinedpolynomials.org/jp.pdf