Articles in Mathematics

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1 Arc Tangent

Introduction

It is shown that the arc tangent is defined by three series in its entire domain.

$$\arctan(x) = -\frac{\pi}{2} + \sum^{0\le i<n} \frac{(-1)^{i+1}}{(2*i+1)*x^{2*i+1}}; \qquad x \le -1 \tag{1}$$

$$\arctan(x) = \sum^{0\le i<n} (-1)^i * \frac{x^{2*i+1}}{2*i+1}; \qquad |x| \le 1 \tag{2}$$

$$\arctan(x) = \frac{\pi}{2} + \sum^{0\le i<n} \frac{(-1)^{i+1}}{(2*i+1)*x^{2*i+1}}; \qquad x \ge 1 \tag{3}$$

The arc tangent is the unknown integral of a rational function.

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} = \frac{1}{x^2+1} \tag{4}$$

Series of Small Values

The initial steps of the division by smallest orders is determined.

$$1/(1+x^2) = 1 - x^2 + x^4 - \frac{x^6}{1+x^2} \tag{5}$$

$$\begin{array}{l} \underline{1+x^2} \\ \quad -x^2 \\ \quad \underline{-x^2-x^4} \\ \qquad\quad x^4 \\ \qquad\quad \underline{x^4+x^6} \\ \qquad\qquad -x^6 \end{array}$$

The division by smallest orders is determined generally.

$$\frac{1}{1+x^2} = \left(\sum^{0\le i<n} (-1)^i * x^{2*i}\right) + \left((-1)^n * \frac{x^{2*n}}{1+x^2}\right) \tag{6}$$

A convergence test is applied.

$$\left|(-1)^i * x^{2*i}\right| > \left|(-1)^{i+1} * x^{2*i+1}\right|; \qquad |x| < 1 \tag{7}$$

The series is integrated without remainder.

$$F(x) = \sum^{0\le i<n} (-1)^i * \frac{x^{2*i+1}}{2*i+1} \tag{8}$$

A convergence test is applied.

$$\left|(-1)^i * \frac{x^{2*i+1}}{2*i+1}\right| > \left|(-1)^{i+1} * \frac{x^{2*(i+1)+1}}{2*(i+1)+1}\right|; \qquad |x| \le 1 \tag{9}$$

The series determines the arc tangent for small values.

$$\arctan(x) = \sum^{0 \le i < n} (-1)^i * \frac{x^{2*i+1}}{2*i+1}; \qquad |x| \le 1 \tag{10}$$

Series of Large Values

The initial steps of the division by highest orders is determined.

$$1/(x^2+1) = \frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{x^6} - \frac{1}{x^6} * \frac{1}{x^2+1} \tag{11}$$

$$\begin{array}{l}
\underline{1 + \frac{1}{x^2}} \\
\quad - \frac{1}{x^2} \\
\quad \underline{-\frac{1}{x^2} - \frac{1}{x^4}} \\
\qquad\qquad \frac{1}{x^4} \\
\qquad\qquad \underline{\frac{1}{x^4} + \frac{1}{x^6}} \\
\qquad\qquad\qquad - \frac{1}{x^6}
\end{array}$$

The division by highest orders is determined generally.

$$\frac{1}{x^2+1} = \left(\sum^{0 < i \le n} \frac{(-1)^i}{x^{2*i}}\right) + \left(\frac{(-1)^n}{x^{2*n} * (x^2+1)}\right) \tag{12}$$

A convergence test is applied.

$$\left|\frac{(-1)^i}{x^{2*i}}\right| > \left|\frac{(-1)^{i+1}}{x^{2*(i+1)}}\right|; \qquad |x| > 1 \tag{13}$$

The series is integrated without remainder.

$$G(x) = A + \sum^{0 \le i < n} \frac{(-1)^{i+1}}{(2*i+1) * x^{2*i+1}} = A + g(x) \tag{14}$$

A convergence test is applied.

$$\left|\frac{(-1)^{i+1}}{(2*i+1) * x^{2*i+1}}\right| > \left|\frac{(-1)^{i+2}}{(2*(i+1)+1) * x^{2*(i+1)+1}}\right|; \qquad |x| \ge 1 \tag{15}$$

The integration constant A is defined by the range of the arc tangent.

$$\lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}; \qquad \lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}; \tag{16}$$

$$\lim_{x\to-\infty} g(x) = 0; \qquad \lim_{x\to\infty} g(x) = 0 \tag{17}$$

Two series of the arc tangent result that differ by the sign of the constant.

$$\arctan(x) = \pm\frac{\pi}{2} + \sum^{0\le i<n} \frac{(-1)^{i+1}}{(2*i+1)*x^{2*i+1}}; \qquad \pm x \ge 1 \tag{18}$$

Summary

Three series determine the arc tangent in its entire domain.

$$\arctan(x) = -\frac{\pi}{2} + \sum^{0\le i<n} \frac{(-1)^{i+1}}{(2*i+1)*x^{2*i+1}}; \qquad x \le -1 \tag{19}$$

$$\arctan(x) = \sum^{0\le i<n} (-1)^i * \frac{x^{2*i+1}}{2*i+1}; \qquad |x| \le 1 \tag{20}$$

$$\arctan(x) = \frac{\pi}{2} + \sum^{0\le i<n} \frac{(-1)^{i+1}}{(2*i+1)*x^{2*i+1}}; \qquad x \ge 1 \tag{21}$$

Both series with positive domains define $\pi/4$.

$$\arctan(1) = \sum^{0\le i<n} \frac{(-1)^{i+1}}{2*i+1} = \frac{\pi}{4} \tag{22}$$

Short cuts exist if the value is not near the bound.

$$\arctan(x) \approx -\frac{\pi}{2} - \frac{1}{x}; \qquad x \ll -1 \tag{23}$$

$$\arctan(x) \approx x; \qquad |x| \ll 1 \tag{24}$$

$$\arctan(x) \approx \frac{\pi}{2} - \frac{1}{x}; \qquad x \gg 1 \tag{25}$$

See [2] for more details.

2 Exponential Function

Introduction

This article determines exponential functions in terms of rational functions and shows that the power of $f(h)$ is an exponential function of single precision according to IEEE 754.

$$f(h) = \frac{120 + 60 * h + 12 * h^2 + h^3}{120 - 60 * h + 12 * h^2 - h^3}; \qquad \lim_{h \to 0} f(h)^{\frac{x}{h}} = \exp(x) \tag{26}$$

The exponential function is defined as the power of the universal constant $\mathtt{e}$ or Euler number.

$$\exp(x) = \mathtt{e}^x \tag{27}$$

Natural logarithm and exponential function are inverse.

$$\ln(\mathtt{e}^x) = x \tag{28}$$

Any other power is determined by the exponential function.

$$a^x = e^{\ln(a) * x} \tag{29}$$

An exact base point is determined.

$$\mathtt{e}^0 = 1 \tag{30}$$

Rational First Degree Extrapolation

A polynomial is determined by three terms.

$$f(h) = a_0 * h^0 + a_1 * h^1 + a_2 * h^2 \tag{31}$$

The first derivative is determined.

$$\frac{df(h)}{dh} = a_1 * h^0 + 2 * a_2 * h^1 \tag{32}$$

The polynomial is determined by three conditions according to the exponential function at two points $h_0 = 0$ and $h_1 = X$.

$$f(0) = 1; \qquad a_0 = 1 \tag{33a}$$

$$\frac{df(0)}{dh} = f(0); \qquad a_1 = a_0 \tag{33b}$$

$$\frac{df(H)}{dh} = f(H); \qquad a_1 * H^0 + 2 * a_2 * H^1 = a_0 * H^0 + a_1 * H^1 + a_2 * H^2 \tag{33c}$$

Each equation is multiplied by a weight w_i. The sum of these weighted equations is determined.

$$w_0 * a_0 + w_1 * a_1 + w_2 * (a_1 + 2 * a_2 * H) = w_0 + w_1 * a_0 + w_2 * \left(a_0 + a_1 * H + a_2 * H^2\right) \tag{34}$$

The expression is rearranged such that all terms of coefficients are grouped on the left hand side.

$$a_0 * (w_0 - w_1 - w_2) + a_1 * (w_1 + w_2 * (1 - H)) + a_2 * \left(w_2 * \left(2 * H - H^2\right)\right) = w_0 \tag{35}$$

The equation equates the polynomial under three conditions.

$$w_0 = f(h); \qquad \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 - H \\ 0 & 0 & 2 * H - H^2 \end{bmatrix} * \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ h \\ h^2 \end{bmatrix} \tag{36}$$

The weight is determined that defines the polynomial.

$$w_0 = \frac{(h+1) * H - h^2 - 2 * h - 2}{H - 2} \tag{37}$$

The equation results a simple rational function if the constant H equals the variable h. The rational function equals the Padé approximant [1/1] under this condition.

$$w_0 = \frac{2 + h}{2 - h} = g(h); \qquad H = h \tag{38}$$

A division of polynomials is applied and results the initial three terms of the exponential series and a remainder.

$$(2 + h)/(2 - h) = 1 + h + \frac{1}{2} * h^2 + \frac{1}{4} * \left(h^3 + \frac{h^4}{2 - h}\right) \approx \mathsf{e}^h \tag{39}$$

The law of exponents applies and results the exponential function if the variable tends to zero.

$$\lim_{h \to 0} \left(\frac{2 + h}{2 - h}\right)^k = \left(\mathsf{e}^h\right)^k = \mathsf{e}^{h*k} = \mathsf{e}^x; \qquad h, k \in \mathbb{R} \tag{40}$$

Rational Extrapolation

A polynomial is determined by $2 * n + 1$ terms.

$$f(h) = \sum^{0 \le i \le 2*n} a_i * h^i \tag{41}$$

As many conditions determine the polynomial.

$$f(0) = 1 \tag{42}$$

$$\frac{d^{i+1} f(0)}{dh^{i+1}} = \frac{d^i f(0)}{dh^i}; \qquad \frac{d^{i+1} f(H)}{dh^{i+1}} = \frac{d^i f(H)}{dh^i}; \qquad 0 \le i < n \tag{43}$$

Each equation is multiplied by a weight w_i. The sum of these weighted equations is determined.

The weights are determined by a system of linear equations.

$$\left[\begin{array}{cccccccc|c} 1 & -1 & -1 & 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 1 & 1-H & -1 & -1 & 0 & 0 & \dots & h \\ 0 & 0 & 2*H-H^2 & 2 & 2-2*H & -2 & -2 & \dots & h^2 \\ 0 & 0 & 3*H^2-H^3 & 0 & 6*H-3*H^2 & 6 & 6-6*H & \dots & h^3 \\ 0 & 0 & 4*H^3-H^4 & 0 & 12*H^2-4*H^3 & 0 & 24*H-12*H^2 & \dots & h^4 \\ 0 & 0 & 5*H^4-H^5 & 0 & 20*H^3-5*H^4 & 0 & 60*H^2-20*H^3 & \dots & h^5 \\ 0 & 0 & 6*H^5-H^6 & 0 & 30*H^4-6*H^5 & 0 & 120*H^3-30*H^4 & \dots & h^6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{array}\right] \tag{44}$$

The polynomial is determined by the zeroth weight and evaluated at $H = h$ where it equals the Padé approximant $[n/n]$.

$$w_0 = \frac{\sum\limits^{0\le i\le n} \frac{(2*n-i)!}{(n-i)!*i!} * h^i}{\sum\limits^{0\le i\le n} (-1)^i * \frac{(2*n-i)!}{(n-i)!*i!} * h^i} = f(h); \qquad H = h \tag{45}$$

The polynomial division by smallest orders results the initial terms of the exponential series and a remainder.

$$f(h) = \sum^{0\le i\le 2*n} \frac{h^i}{i!} + \mathcal{O}\left(h^{2*n}\right) \tag{46}$$

The law of exponents applies and results the exponential function if the variable tends to zero.

$$\lim_{h\to 0} (w_0)^k = \left(\mathsf{e}^h\right)^k = \mathsf{e}^{h*k} = \mathsf{e}^x; \qquad h, k \in \mathbb{R} \tag{47}$$

Exponential Function of Single Precision

An exponential function of single precision according to IEEE 754 is determined by a rational function that equals the Padé approximant [3/3].

$$f(h) = \frac{\sum\limits^{0\le i\le 3} \frac{(6-i)!}{(3-i)!*i!} * h^i}{\sum\limits^{0\le i\le 3} (-1)^i * \frac{(6-i)!}{(3-i)!*i!} * h^i} = \frac{120 + 60*h + 12*h^2 + h^3}{120 - 60*h + 12*h^2 - h^3} \tag{48}$$

The value is computed by law of exponents with $h = 0.1$.

$$(f(0.1))^k \approx \mathsf{e}^{k*0.1} = \mathsf{e}^x \tag{49}$$

The polynomial division by smallest orders is determined in order to estimate the maximum error.

$$f(h) = \left(\sum^{0\le i<7} \frac{h^i}{i!}\right) + \frac{h^7}{4800} + \frac{h^8}{28800} + \mathcal{O}\left(h^9\right) \tag{50}$$

$$= \left(\sum^{0\le i\le 7} \frac{h^i}{i!}\right) + \frac{h^7}{100800} + \frac{h^8}{28800} + \mathcal{O}\left(h^9\right) \tag{51}$$

The maximum error is estimated by the remainder compared to the single extrapolation.

$$e(h) = \left|\frac{2*h^7}{100800}\right| + \left|\frac{2*h^8}{28800}\right|; \qquad e(0.1) < 2.7*10^{-12} \tag{52}$$

The range of single precision is about $\pm 3.403 * 10^{38}$ with seven significant leading digits. The domain of the extrapolation is determined.

$$|x| = \ln\left(3.403 * 10^{38}\right) < 90 \tag{53}$$

Factor k is separated into a binary number. A maximum of nine multiplications are required for the domain of single precision and a step h.

$$90 = 900 * 0.1 < 1024 * 0.1 = 2^{10} * 0.1 \tag{54}$$

The precision of computers is finite and usually half a bit of precision is lost for each multiplication. A maximum of two multiplications is required for each binary part. Therefore a maximum of four bits of precision is lost if double precision is used for computation.

$$\log_2\left(2*9*\frac{1}{2}\right) < 4 \tag{55}$$

See [2] for more details.

Listing 1: e-function of single precision in C

```
#include <math.h>
#include <stdio.h>
#include <stdlib.h>

static double wexp1n3(double const x)
{
  double const xx = x*x;
  double const A = 120.l + 12.l*xx;
  double const B = x*(60.l + xx);
  return (A+B)/(A-B);
}

double exp1(double const x)
{
  unsigned j, i; // unsigned suffices for h=0.1 and LDBL_MAX
  double wj, factor;
  // compute exponent and initial factor............................
  j = (unsigned)(fabs(x)/0.1l) + 1; // |x|/max(h)
  factor = wexp1n3(x/j); // Gewicht von x/j
  // compute power ..............................................
  wj = j&1 ? factor : 1.l; // begin with w^1 or w^0
  for(i = 2; i <= j; i <<= 1) // all exponents 2,4,8,16 <=j
  {
    factor *= factor; // w^i
    if(j&i) // if i is part of j
    {
      wj *= factor;
    }
  }
  return wj;
}

int main(int argc, char ** argv)
{
  double x, e, en;
  if(argc != 2)
  {
    fprintf(stderr, "%s x\n", argv[0]);
    exit(1);
  }
  x = atof(argv[1]);
  en = exp1(x);
  e = exp(x);
  printf("expn(%lf)=%.20lg\n", x, exp1(x));
  printf("exp (%lf)=%.20lg\n", x, exp(x));
  printf("fehler~%lg\n", (en-e)/e);
  return 0;
}
```

3 Lagrange's Interpolation Formula

Lagrange's Interpolation Formula is determined as a special case of polynomial transposition [2].

A number of points is determined with unique locations x_j.

$$y_j = f(x_j); \qquad 0 \leq j < n \tag{56}$$

Therefore an interpolation polynomial is determined by as many terms.

$$y = f(x) = \sum^{0 \leq i < n} a_i * x^i \tag{57}$$

Every point is assigned a base polynomial or weight w_j. Suppose the sum of all weighted conditions equals the polynomial.

$$f(x) = \sum^{0 \leq i < n} a_i * x^i = \sum^{0 \leq j < n} w_j * y_j = \sum^{0 \leq j < n} w_j * \sum^{0 \leq i < n} a_i * x_j^i \tag{58}$$

The double sum is interchanged.

$$f(x) = \sum^{0 \leq i < n} a_i * x^i = \sum^{0 \leq j < n} w_j * y_j = \sum^{0 \leq i < n} a_i * \sum^{0 \leq j < n} w_j * x_j^i \tag{59}$$

The base polynomials are determined by a system of linear equations according to a comparison by coefficients.

$$\sum^{0 \leq j < n} w_j * x_j^i = x^i; \qquad 0 \leq i < n \tag{60}$$

The base matrix is a transposed Vandermonde matrix.

$$G = \sum^{0 \leq i < n} \left\langle \sum^{0 \leq j < n} \langle x_j^i \rangle \right\rangle \tag{61}$$

The determinant of a Vandermonde matrix equals the product of all possible differences. The determinant is non-zero if all locations are unique.

$$\det(G) = \prod^{1 \leq i < n} \prod^{0 \leq j < i} (x_i - x_j) \tag{62}$$

A base polynomial is determined by Cramer's rule. Thus a source matrix is a variant of the base matrix for which one column is replaced by the source. The determinant of a source matrix is determined accordingly.

$$\det(Q_m) = \prod^{1 \leq i < n} \prod^{0 \leq j < i} \begin{cases} x - x_j, & \text{if } i = m \\ x_i - x, & \text{if } j = m \\ x_i - x_j, & \text{otherwise} \end{cases} \tag{63}$$

A base polynomial is determined by Cramer's rule. A number of differences and signs cancel.

$$w_j = \frac{\det(Q_j)}{\det(G)} = \frac{\prod\limits_{\substack{i \neq j}}^{0 \le i < n} (x_i - x)}{\prod\limits_{\substack{i \neq j}}^{0 \le i < n} (x_i - x_j)} \tag{64}$$

Lagrange's Interpolation formula is determined by polynomial transposition.

$$f(x) = \sum^{0 \le j < n} w_j * y_j \tag{65}$$

4 Logarithm

A conditionally convergent series of the natural logarithm is derived for its entire domain.

The natural logarithm is the unknown integral of a hyperbola.

$$y = \ln(x); \qquad \frac{d}{dx}\ln(x) = \frac{1}{x}; \qquad x > 0 \tag{66}$$

Derivatives of higher order follow accordingly.

$$\frac{d^j}{dx^j}\ln(x) = (-1)^{j-1} * \frac{(j-1)!}{U^j} \tag{67}$$

Natural logarithm and exponential function are inverse.

$$\ln(\mathtt{e}^x) = x \tag{68}$$

Logarithms of another base than e are multiples of the natural logarithm.

$$b^y = x; \qquad y = \log_b(x) = \frac{\ln(x)}{\ln(b)} \tag{69}$$

The logarithm is approximated by a polynomial.

$$f(x) = \sum^{0\leq i<n} a_i * x^i \tag{70}$$

The polynomial is to equate a point of the logarithm and a number of derivatives at that point.

$$f(U) = \frac{d^0 f(U)}{dx^0} = \ln(U) = Y; \qquad \frac{d^j f(U)}{dx^j} = (-1)^{j-1} * \frac{(j-1)!}{U^j}; \qquad j > 0 \tag{71}$$

Each condition is scaled by a weight w_i. A sum of all weighted conditions is determined.

$$w_0 * f(U) + \sum^{1\leq j<n} w_j * \frac{d^j f(U)}{dx^j} = w_0 * Y + \sum^{1\leq j<n} w_j * (-1)^{j-1} * \frac{(j-1)!}{U^j} \tag{72}$$

Suppose the weighted sum equals the polynomial.

$$f(x) = w_0 * f(U) + \sum^{1\leq j<n} w_j * \frac{d^j f(U)}{dx^j} \tag{73}$$

The derivatives of the polynomial are determined at the base point.

$$f(x) = a_0 + a_1 * x + a_2 * x^2 + a_3 * x^3 + a_4 * x^4 + a_5 * x^5 + ... \tag{74}$$

$$\frac{df(U)}{dx} = a_1 + 2 * a_2 * U + 3 * a_3 * U^2 + 4 * a_4 * U^3 + 5 * a_5 * U^4 + ... \tag{75}$$

$$\frac{d^2 f(U)}{dx^2} = 2 * a_2 + 6 * a_3 * U + 12 * a_4 * U^2 + 20 * a_5 * U^3 + ... \tag{76}$$

$$\frac{d^3 f(U)}{dx^3} = 6 * a_3 + 24 * a_4 * U + 60 * a_5 * U^2 + ... \tag{77}$$

$$... \tag{78}$$

The descending faculty is defined in order to express derivatives generally.

$$(a¡b) = \frac{a!}{(a-b)!} \tag{79}$$

A derivative of the polynomial is defined generally.

$$\frac{d^j f(U)}{dx^j} = \sum^{j \le i < n} a_i * (i¡j) * U^{i-j} \tag{80}$$

The weights are determined by a system of linear equations according to a comparison by the coefficients a_i.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ U & 1 & 0 & 0 & \dots \\ U^2 & 2*U & 2 & 0 & \dots \\ U^3 & 3*U^2 & 6*U & 6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} * \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \end{bmatrix} \tag{81}$$

The base matrix is triangular such that the solution is available explicitly.

$$w_0 = 1 \tag{82}$$

$$w_1 = x - U * w_0 \tag{83}$$

$$w_2 = \frac{1}{2} * \left(x^2 - U^2 * w_0 - 2 * U * w_1\right) \tag{84}$$

$$w_3 = \frac{1}{6} * \left(x^3 - U^3 * w_0 - 3 * U^2 * w_1 - 6 * U * w_2\right) \tag{85}$$

$$w_m = \frac{1}{m!} * \left(x^m - \sum^{0 \le k < m} (m¡k) * U^{m-k} * w_k\right) \tag{86}$$

The weights are noted explicitly.

$$w_0 = 1 \tag{87}$$

$$w_1 = x - U \tag{88}$$

$$w_2 = \frac{1}{2!} * \left(x^2 - U^2 - 2 * U * (x - U)\right) = \frac{1}{2!} * (x - U)^2 \tag{89}$$

$$w_3 = \frac{1}{3!} * \left(x^3 - U^3 - 3 * U^2 * (x - U) - \frac{(3¡2)}{2!} * U * (x - U)^2\right) = \frac{1}{3!} * (x - U)^3 \tag{90}$$

$$w_m = \frac{1}{m!} * \left(x^m - \sum^{0 \le k < m} \binom{m}{k} * U^{m-k} * (x - U)^k\right) = \frac{1}{m!} * (x - U)^m \tag{91}$$

The value of the weights is substituted into the polynomial.

$$f(x) = f(U) + \sum^{1 \le j < n} \frac{1}{j!} * (x - U)^j * \frac{d^j f(U)}{dx^j} \tag{92}$$

The derivatives of the logarithm are noted explicitly.

$$f(x) = Y + \sum^{1 \le j < n} \frac{1}{j!} * (x - U)^j * (-1)^{j-1} * \frac{(j-1)!}{U^j} \tag{93}$$

A series is determined.

$$f(x) = Y + \sum^{1 \leq j < n} (-1)^{j-1} * \frac{(x-U)^j}{j * U^j} \tag{94}$$

D'Alembert's convergence test of 1/2 is applied.

$$\frac{1}{2} * \left| \frac{(x-U)^j}{j * U^j} \right| > \left| \frac{(x-U)^{j+1}}{(j+1) * U^{j+1}} \right| \tag{95a}$$

$$\left| \frac{j+1}{j} * U \right| > 2 * |x-U| \tag{95b}$$

$$|U| > 2 * |x-U| \tag{95c}$$

The series converges conditionally.

$$\ln(x) = Y + \sum^{1 \leq j < n} (-1)^{j-1} * \frac{(x-U)^j}{j * U^j}; \qquad U > 0; \quad U > 2 * |x-U|; \quad Y = \ln(U) \tag{96}$$

Base points may be determined by the exponential function.

$$\mathsf{e}^2 > 2 * \left|10 - \mathsf{e}^2\right|; \qquad f(10) \approx 2 + \sum^{1 \leq j < 3} (-1)^{j-1} * \frac{\left(10 - \mathsf{e}^2\right)^j}{j * \mathsf{e}^{2*j}} \approx 2.305630 \tag{97}$$

$$\mathsf{e}^4 > 2 * \left|50 - \mathsf{e}^4\right|; \qquad f(50) \approx 4 + \sum^{1 \leq j < 3} (-1)^{j-1} * \frac{\left(50 - \mathsf{e}^4\right)^j}{j * \mathsf{e}^{4*j}} \approx 3.912036 \tag{98}$$

5 Poisson's Equation

Introduction

A solution to Poisson's Equation of one dimension is presented.

$$\frac{df(x)}{dx} = \text{const} \tag{99}$$

The domain is discretized by a number of equidistant points.

$$y_i = f(x_i); \qquad \frac{df(x_i)}{dx} = s_i \tag{100}$$

Poisson Operator

A local polynomial is assigned to each point.

$$f(h) = a^0 + a^1 * h + a^2 * h^2 \tag{101}$$

Poisson's equation is applied to each polynomial.

$$2 * a_2 = s_i \tag{102}$$

Adjacent polynomials are joined by Dirichlet conditions.

$$y_{i-1} = f(-h) = a^0 - a^1 * h + a^2 * h^2 \tag{103}$$
$$y_{i+1} = f(h) = a^0 + a^1 * h + a^2 * h^2 \tag{104}$$

The operator is determined by a transposition [2].

$$y_i = f(0) = w_0 * y_{i-1} + w_1 * s_i + w_2 * y_{i+1} \tag{105}$$

The weights are determined by a system of linear equations.

$$\begin{bmatrix} 1 & 0 & 1 \\ -h & 0 & h \\ h^2 & 2 & h^2 \end{bmatrix} * \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \qquad w = \left(\frac{1}{2}, -\frac{h^2}{2}, \frac{1}{2}\right) \tag{106}$$

A value is determined explicitly by a transposed local polynomial.

$$y_i = \frac{1}{2} * y_{i-1} - \frac{h^2}{2} * q_i + \frac{1}{2} * y_{i+1} \tag{107}$$

A value is determined implicitly by a Poisson Operator.

$$-y_{i-1} + 2 * y_i - y_{i+1} = -h^2 * s_i = q_i \tag{108}$$

System of Equations

A uniform tridiagonal square system of equations is determined by n equidistant base points. The bounds of the domain are contained in the sources.

$$D_n * y = \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & & . \quad . \quad . & & & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{bmatrix} * \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ . \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ . \\ q_{n-4} \\ q_{n-3} \\ q_{n-2} \end{bmatrix} \tag{109}$$

The determinant of a domain of n points is determined. The system consists of $n-2$ equations.

$$\det(D_n) = d_n = n - 1 \tag{110}$$

A source matrix of $n-2$ equations and the k-th column replaced is determined.

$$P_{n,k} = \begin{bmatrix} 2 & -1 & & & q_1 & & & & \\ -1 & 2 & -1 & & q_2 & & & & \\ & -1 & 2 & -1 & q_3 & & & & \\ & & & . & . & . & & & \\ & & & & q_{n-4} & -1 & 2 & -1 & \\ & & & & q_{n-3} & & -1 & 2 & -1 \\ & & & & q_{n-2} & & & -1 & 2 \end{bmatrix} \tag{111}$$

The determinant of a source matrix is determined.

$$\det(P_{n,k}) = p_{n,k} = d_{n-1-k} * \sum^{0 \le i < k} (q_{i+1} * D_{i+2}) + d_{k+2} * \sum^{k \le i < n-1} (q_{i+1} * D_{n-1-i}) \tag{112}$$

The solution to Poisson's equation is determined by Cramer's rule.

$$y_{k+1} = \frac{p_{n,k}}{d_n} = \frac{(n-2-k) * \sum^{0 \le i < k} (q_{i+1} * (i+1)) + (k+1) * \sum^{k \le i < n-2} (q_{i+1} * (n-2-i))}{n-1} \tag{113}$$

The solution to Laplace's equation is determined by the sources at the ends only.

$$y_{k+1} = \frac{p_{n,k}}{d_n} = \frac{(n-2-k) * q_1 + (k+1) * q_{n-2}}{n-1} \tag{114}$$

See [2] for an interpolation of the sine by this same method.

6 Polynomial Integration

Introduction

This articles proofs that $F(x)$ is an integration of the polynomial $f(x)$ for a change v of any size.

$$f(x) = \sum^{0 \le i < n} a_i * x^i; \qquad F(x) = C + \sum^{0 \le i < n} a_i * \frac{x^{i+1}}{i+1} \tag{115}$$

The differentiation of a polynomial is discussed before its integration.

Differentiation

The variable is separated.

$$x = u + v \tag{116}$$

The separation is substituted.

$$f(u+v) = \sum^{0 \le i < n} a_i * (u+v)^i = g(u,v) \tag{117}$$

The binomial expansion is applied by v.

$$g(u,v) = \sum^{0 \le i < n} a_i * \sum^{0 \le j \le i} \binom{i}{j} * u^{i-j} * v^j \tag{118}$$

The descending faculty is defined in order to dissolve the binomial coefficient.

$$(i¡j) = \frac{i!}{(i-j)!}; \qquad (i¡0) = 1 \tag{119}$$

The sums are transposed.

$$g(u,v) = \sum^{0 \le j < n} \frac{v^j}{j!} * \sum^{j \le i < n} a_i * (i¡j) * u^{i-j} \tag{120}$$

A derivative is defined by the value of the inner sum.

$$\frac{d^j f(u)}{du^j} = \sum^{j \le i < n} a_i * (i¡j) * u^{i-j} \tag{121}$$

The Taylor series of the polynomial is determined.

$$g(u,v) = \sum^{0 \le j < n} \frac{v^j}{j!} * \frac{d^j f(u)}{du^j} \tag{122}$$

Integration

The integration to the next degree is defined by an offset and a constant.

$$h(u,v) = c + \sum^{0\le j<n} \frac{v^{j+1}}{(j+1)!} * \frac{d^j f(u)}{du^j}; \qquad \frac{d^j f(u)}{du^j} = \frac{d^{j+1}F(u)}{du^{j+1}} \tag{123}$$

The derivatives are given explicitly.

$$h(u,v) = c + \sum^{0\le j<n} \frac{v^{j+1}}{(j+1)!} * \sum^{j\le i<n} a_i * (i\text{¡}j) * u^{i-j} \tag{124}$$

The sums are transposed.

$$h(u,v) = c + \sum^{0\le i<n} a_i * \sum^{i\le j<n} \frac{v^{j+1}}{(j+1)!} * (i\text{¡}j) * u^{i-j} \tag{125}$$

The separation of the variable cannot be reversed due to the offset. Therefore the integration is developed at a constant location U.

$$x = u + v; \qquad u = U = \text{const}; \qquad v = x - U \tag{126}$$

The definitions are substituted and another function is determined.

$$h(U, x-U) = c + \sum^{0\le i<n} a_i * \sum^{i\le j<n} \frac{(x-U)^{j+1}}{(j+1)!} * (i\text{¡}j) * U^{i-j} = F(x) \tag{127}$$

The equation is rearranged.

$$F(x) = c + \sum^{0\le i<n} a_i * \sum^{i\le j<n} \frac{(i\text{¡}j)}{(j+1)!} * U^{i-j} * (x-U)^{j+1} \tag{128}$$

The binomial is expanded.

$$F(x) = c + \sum^{0\le i<n} a_i * \sum^{i\le j<n} \frac{(i\text{¡}j)}{(j+1)!} * U^{i-j} * \sum^{0\le k\le j+1} \binom{j+1}{k} * x^k * (-U)^{j+1-k} \tag{129}$$

The inner sums are transposed in order to group all constants.

$$F(x) = c + \sum^{0\le i<n} a_i * \sum^{0\le k\le i+1} x^k * \sum_{j+1\ge k}^{0\le j\le i} \frac{(i\text{¡}j)}{(j+1)!} * U^{i-j} * \binom{j+1}{k} * (-U)^{j+1-k} \tag{130}$$

An identity is required to rearranged the coefficients.

$$\binom{C}{B} * \binom{B}{A} = \frac{(C\text{¡}A) * ((C-A)\text{¡}(B-A))}{(B-A)! * (B\text{¡}A)} * \frac{(B\text{¡}A)}{A!} = \binom{C}{A} * \binom{C-A}{B-A} \tag{131}$$

The coefficients are rearranged such that only one factor depends on j.

$$\frac{(i¡j)}{(j+1)!} * \binom{j+1}{k} = \frac{((i+1)¡(j+1))}{(i+1)*(j+1)!} * \binom{j+1}{k} \tag{132}$$

$$= \frac{1}{i+1} * \binom{i+1}{j+1} * \binom{j+1}{k} \tag{133}$$

$$= \frac{1}{i+1} * \binom{i+1}{k} * \binom{i+1-k}{j+1-k} \tag{134}$$

These coefficients are substituted. The exponent of the constant simplifies.

$$F(x) = c + \sum^{0 \le i < n} a_i * \sum^{0 \le k \le i+1} \frac{U^{i+1-k}}{i+1} * \binom{i+1}{k} * x^k * \sum_{j+1 \ge k}^{0 \le j \le i} (-1)^{j+1-k} * \binom{i+1-k}{j+1-k} \tag{135}$$

Pascal's triangle is defined with alternating signs in order to simplify the bounds of the sums.

$$\begin{array}{ccccccccc|c} & & & & 1 & & & & & = 1 \\ & & & 1 & - & 1 & & & & = 0 \\ & & 1 & - & 2 & + & 1 & & & = 0 \\ & 1 & - & 3 & + & 3 & - & 1 & & = 0 \\ 1 & - & 4 & + & 6 & - & 4 & + & 1 & = 0 \\ & & & . & . & . & & & & = 0 \end{array} \tag{136}$$

The zeroth line of Pascal's triangle follows under two conditions.

$$i = j; \quad i+1 = k; \qquad \frac{U^{i+1-k}}{i+1} * \binom{i+1}{k} = \frac{1}{k}; \quad (-1)^{j+1-k} * \binom{i+1-k}{j+1-k} = 1 \tag{137}$$

This case results once for each term of the integrated polynomial is noted exclusively.

$$F(x) = c + \sum^{0 \le i < n} a_i * \sum^{0 \le k < i+1} \frac{U^{i+1-k}}{i+1} * \binom{i+1}{k} * x^k * \sum_{j+1 \ge k}^{0 \le j \le i} (-1)^{j+1-k} * \binom{i+1-k}{j+1-k} \tag{138a}$$

$$+ \sum^{0 \le i < n} a_i * \frac{x^{a+1}}{i+1} \tag{138b}$$

The value of the innermost sum cancels if it maps to a full line of Pascal's triangle with alternating signs.

$$1 \le k < i+1; \qquad \sum_{j+1 \ge k}^{0 \le j \le i} (-1)^{j+1-k} * \binom{i+1-k}{j+1-k} = 0 \tag{139}$$

Therefore the bound of the middle sum simplifies to one index $k = 0$.

$$F(x) = c + \sum^{0 \le i < n} a_i * \frac{U^{i+1}}{i+1} * \sum^{0 \le j \le i} (-1)^{j+1} * \binom{i+1}{j+1} + \sum^{0 \le i < n} a_i * \frac{x^{i+1}}{i+1} \tag{140}$$

The inner sum maps to a line of Pascal's triangle with alterning signs without its zeroth element.

$$\sum^{0 \le j \le i} (-1)^{j+1} * \binom{i+1}{j+1} = -1 \tag{141}$$

The integration polynomial is determined by simple sums.

$$F(x) = c - \sum^{0\leq i<n} a_i * \frac{U^{i+1}}{i+1} + \sum^{0\leq i<n} a_i * \frac{x^{i+1}}{i+1} = C + \sum^{0\leq i<n} a_i * \frac{x^{i+1}}{i+1} \tag{142}$$

The integration polynomial simplifies if the constant equals the origin.

$$U = 0; \qquad F(x) = c + \sum^{0\leq i<n} a_i * \frac{x^{i+1}}{i+1} \tag{143}$$

See [2] for integration of polynomials of any dimension and offset.

7 Sine Theorem

Introduction

This article shows how to express the sine exactly as a sum along the components of a Fibonacci number. The sum is derived by a recurrence relation on a sine operator.

Sine Operator

The sine operator is determined by two base points one left to and another at the origin and one condition of simple harmonic motion of a distribution coefficient c at the origin.

$$f(-H) = y_L; \qquad f(0) = y_0; \qquad c^2 * f(0) + \frac{d^2 f(0)}{dh^2} = 0; \qquad c > 0 \tag{144}$$

Three conditions determine a polynomial of three terms.

$$f(h) = a_0 * h^0 + a_1 * h^1 + a_2 * h^2; \qquad \frac{d^2 f(h)}{dh^2} = 2 * a_2 \tag{145}$$

Each condition is scaled by a weight w_i. A sum of the weighted conditions is determined.

$$w_L * \left(a_0 - a_1 * H + a_2 * H^2\right) + w_0 * a_0 + w_1 * \left(c^2 * a_0 + 2 * a_2\right) = w_L * y_L + w_0 * y_0 \tag{146}$$

The sum equals the polynomial under three conditions.

$$f(h) = w_L * y_L + w_0 * y_0; \qquad \begin{bmatrix} 1 & 1 & c^2 \\ -H & 0 & 0 \\ H^2 & 0 & 2 \end{bmatrix} * \begin{bmatrix} w_L \\ w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} 1 \\ h \\ h^2 \end{bmatrix} \tag{147}$$

The polynomial is determined by two weights.

$$w_L = -\frac{h}{H}; \qquad w_0 = -\frac{c^2 * h * H * (H + h) - 2 * H - 2 * h}{2 * H} \tag{148}$$

The solution simplifies if the constant equals the variable that is the distance H to the left equals the extrapolation to the right.

$$w_L = -1; \qquad w_0 = 2 - c^2 * h^2 \tag{149}$$

Analysis

Suppose the solution is a sine of a frequency d.

$$f(h) = R * \sin(\varphi + d * h) \tag{150}$$

The values and weights are substituted into the polynomial.

$$f(h) = w_L * y_L + w_0 * y_0 \tag{151}$$

$$R * \sin(\varphi + d * h) = R * \sin(\varphi - d * h) * w_L + R * \sin(\varphi) * w_0 \tag{152}$$

$$R * \sin(\varphi + d * h) = R * \sin(\varphi - d * h) * (-1) + R * \sin(\varphi) * \left(2 - c^2 * h^2\right) \tag{153}$$

A trigonometric addition formula applies.

$$\sin(a \pm b) = \sin(a) * \cos(b) \pm \cos(a) * \sin(b) \tag{154}$$

The formula is applied and two terms cancel. The scalar $R * \sin(\varphi)$ cancels. Note that w_L equals negative One.

$$\cos(d * h) = -\cos(d * h) + \left(2 - c^2 * h^2\right) \tag{155}$$

Distribution coefficient c and frequency d are not equal. However, the limit of the right hand side tends to c for small differences, see [2] for details.

$$d = \frac{1}{h} * \arccos\left(1 - \frac{1}{2} * c^2 * h^2\right); \qquad \lim_{h \to 0}\left(\frac{1}{h} * \arccos\left(1 - \frac{1}{2} * c^2 * h^2\right)\right) = c \tag{156}$$

The upper bound of difference h is determined by the domain of the arcus cosine.

$$\left|1 - \frac{1}{2} * c^2 * h^2\right| \leq 1; \qquad c^2 * h^2 \leq 2 \tag{157}$$

The polynomial is determined only if h is non-zero. Therefore the lower bound is excluded. The value of $\arccos(1)$ is zero and would result a difference of zero. Therefore the upper bound is excluded. The intersected domain is determined.

$$0 < h < \frac{\sqrt{2}}{c} \tag{158}$$

Sine Recurrence Relation

The sine recurrence relation is a numerical pattern that determines the sine. Values are repeatedly determined by two preceding values. These values are scaled by the same weights due to a uniform discretization.

$$\begin{aligned}
y_h &= y_L * w_L + y_0 * w_0 && (159)\\
y_{2h} &= y_0 * w_L + y_h * w_0 && (160)\\
&= y_0 * w_L + (y_L * w_L + y_0 * w_0) * w_0 && (161)\\
&= y_L * w_L * w_0 + y_0 * \left(w_L + w_0^2\right) && (162)\\
y_{3h} &= y_h * w_L + y_2 * h * w_0 && (163)\\
&= y_L * \left(w_L^2 + w_L * w_0^2\right) + y_0 * \left(2 * w_L * w_0 + w_0^3\right) && (164)\\
&\dots
\end{aligned}$$

Each value of the right-hand-side is scaled by a composed weight in terms of a sum. The sum is similar to a binomial expansion but does not reduce to a basic operation.

$$\begin{aligned}
W_{j,k} &= \sum^{0 \leq i \leq \lfloor \frac{j-k}{2} \rfloor} \binom{j-k-i}{i} * w_L^{i+k} * w_0^{j-k-2*i} && (165)\\
&= \sum^{0 \leq i \leq \lfloor \frac{j-k}{2} \rfloor} (-1)^{i+k} * \binom{j-k-i}{i} * w_0^{j-k-2*i} && (166)
\end{aligned}$$

The j-th value of the repetition is determined.

$$y_{j*h} = y_L * W_{j,1} + y_0 * W_{j,0} \tag{167}$$

The value at the origin y_0 equals zero such that the value to the right y_1 depends only on the value to the left y_L.

$$y_{j*h} = y_L * \sum^{0 \leq i \leq \lfloor \frac{j-1}{2} \rfloor} (-1)^{i+1} * \binom{j-1-i}{i} * w_0^{j-1-2*i} \tag{168}$$

The offset φ equals zero since y_0 equals zero. An offset of π or 180 deg is determined by the sign of y_L.

$$y_{j*h} = y_L * W_{j,1} \tag{169}$$

$$\sin(j*d*h) = \sin(-d*h) * W_{j,1} \tag{170}$$

The sine theorem is determined.

$$\frac{\sin(j*d*h)}{\sin(-d*h)} = \sum^{0 \leq i \leq \lfloor \frac{j-1}{2} \rfloor} (-1)^{i+1} * \binom{j-1-i}{i} * w_0^{j-1-2*i} \tag{171}$$

$$\frac{\sin(j*d*h)}{\sin(d*h)} = \sum^{0 \leq i \leq \lfloor \frac{j-1}{2} \rfloor} (-1)^{i} * \binom{j-1-i}{i} * w_0^{j-1-2*i} \tag{172}$$

The Fibonacci recurrence relation is a special case of the sine recurrence relation with weights w_L and w_0 of identity.

$$F_{j+2} = F_{j+1} + F_j \tag{173}$$

A Fibonacci number F is the sum of the binomial coefficients only.

$$F_j = \sum^{0 \leq i \leq \lfloor \frac{j-1}{2} \rfloor} \binom{j-1-i}{i} \tag{174}$$

Listing 2: sine theorem in C with gmp [3]

```
#include <assert.h>
#include <math.h>
#include <stdio.h>
#include <gmp.h>

void gbinom(mpf_t r, unsigned const a, unsigned const b)
{
  unsigned i;
  mpf_set_ui(r, 1);
  for(i = 1; i <= b; ++i)
  {
    mpf_mul_ui(r, r, a-i+1);
    mpf_div_ui(r, r, i);
  }
}

int main(void)
{
  double const c = 3.,h = .2, w0 = 2.-h*h*c*c;
  double const d = acos(w0/2.)/h, r = 1./sin(d*h);
  unsigned i, j;
  double s;
  mpf_t e, b, t;
  FILE * f = fopen("gsine.dat", "w");
  assert(f);
  mpf_set_default_prec(1024);
  mpf_init(b); mpf_init(e); mpf_init(t);
  for(j = 1; j < 500; ++j)
  {
    mpf_set_ui(e, 0);
    for(i = 0; 2*i <= j-1; ++i)
    {
      gbinom(b, j-i-1, i);
      if(i%2) { mpf_neg(b, b); }
      mpf_set_d(t, w0);
      mpf_pow_ui(t, t, j-1-2*i);
      mpf_mul(b, b, t);
      mpf_add(e, e, b);
    }
    s = sin(d*(j)*h)*r;
    fprintf(f, "%f %f %f %.24f\n", j*h, mpf_get_d(e), s, mpf_get_d(e)-s);
  }
  fclose(f);
  mpf_clear(b); mpf_clear(e); mpf_clear(t);
  printf("%f*sin(%f*x)\n", r, d);
  return 0;
}
```

References

[1] Articles in Mathematics, Hans-Dieter Reuter, http://www.joinedpolynomials.org/aim.pdf

[2] Joined Polynomials, Hans-Dieter Reuter, http://www.joinedpolynomials.org/jp.pdf

[3] http://gmplib.org, visited 15. October 2010