Exponential Function

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Introduction

This article determines exponential functions in terms of rational functions and shows that the power of $f(h)$ is an exponential function of single precision according to IEEE 754.

$$f(h) = \frac{120 + 60 * h + 12 * h^2 + h^3}{120 - 60 * h + 12 * h^2 - h^3}; \qquad \lim_{h \to 0} f(h)^{\frac{x}{h}} = \exp(x) \tag{1}$$

The exponential function is defined as the power of the universal constant e or Euler number.

$$\exp(x) = \mathsf{e}^x \tag{2}$$

Natural logarithm and exponential function are inverse.

$$\ln(\mathsf{e}^x) = x \tag{3}$$

Any other power is determined by the exponential function.

$$a^x = e^{\ln(a) * x} \tag{4}$$

An exact base point is determined.

$$\mathsf{e}^0 = 1 \tag{5}$$

Rational First Degree Extrapolation

A polynomial is determined by three terms.

$$f(h) = a_0 * h^0 + a_1 * h^1 + a_2 * h^2 \tag{6}$$

The first derivative is determined.

$$\frac{df(h)}{dh} = a_1 * h^0 + 2 * a_2 * h^1 \tag{7}$$

The polynomial is determined by three conditions according to the exponential function at two points $h_0 = 0$ and $h_1 = X$.

$$f(0) = 1; \qquad a_0 = 1 \tag{8a}$$

$$\frac{df(0)}{dh} = f(0); \qquad a_1 = a_0 \tag{8b}$$

$$\frac{df(H)}{dh} = f(H); \qquad a_1 * H^0 + 2 * a_2 * H^1 = a_0 * H^0 + a_1 * H^1 + a_2 * H^2 \tag{8c}$$

Each equation is multiplied by a weight w_i. The sum of these weighted equations is determined.

$$w_0 * a_0 + w_1 * a_1 + w_2 * (a_1 + 2 * a_2 * H) = w_0 + w_1 * a_0 + w_2 * \left(a_0 + a_1 * H + a_2 * H^2\right) \tag{9}$$

The expression is rearranged such that all terms of coefficients are grouped on the left hand side.

$$a_0 * (w_0 - w_1 - w_2) + a_1 * (w_1 + w_2 * (1 - H)) + a_2 * \left(w_2 * \left(2 * H - H^2\right)\right) = w_0 \tag{10}$$

The equation equates the polynomial under three conditions.

$$w_0 = f(h); \qquad \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 - H \\ 0 & 0 & 2 * H - H^2 \end{bmatrix} * \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ h \\ h^2 \end{bmatrix} \tag{11}$$

The weight is determined that defines the polynomial.

$$w_0 = \frac{(h + 1) * H - h^2 - 2 * h - 2}{H - 2} \tag{12}$$

The equation results a simple rational function if the constant H equals the variable h. The rational function equals the Padé approximant [1/1] under this condition.

$$w_0 = \frac{2 + h}{2 - h} = g(h); \qquad H = h \tag{13}$$

A division of polynomials is applied and results the initial three terms of the exponential series and a remainder.

$$(2 + h)/(2 - h) = 1 + h + \frac{1}{2} * h^2 + \frac{1}{4} * \left(h^3 + \frac{h^4}{2 - h}\right) \approx \mathsf{e}^h \tag{14}$$

The law of exponents applies and results the exponential function if the variable tends to zero.

$$\lim_{h \to 0} \left(\frac{2 + h}{2 - h}\right)^k = \left(\mathsf{e}^h\right)^k = \mathsf{e}^{h*k} = \mathsf{e}^x; \qquad h, k \in \mathbb{R} \tag{15}$$

Rational Extrapolation

A polynomial is determined by $2 * n + 1$ terms.

$$f(h) = \sum^{0 \le i \le 2*n} a_i * h^i \tag{16}$$

As many conditions determine the polynomial.

$$f(0) = 1 \tag{17}$$

$$\frac{d^{i+1}f(0)}{dh^{i+1}} = \frac{d^{i}f(0)}{dh^{i}}; \qquad \frac{d^{i+1}f(H)}{dh^{i+1}} = \frac{d^{i}f(H)}{dh^{i}}; \qquad 0 \le i < n \tag{18}$$

Each equation is multiplied by a weight w_i. The sum of these weighted equations is determined. The weights are determined by a system of linear equations.

$$\left[\begin{array}{cccccccc|c} 1 & -1 & -1 & 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 1 & 1-H & -1 & -1 & 0 & 0 & \dots & h \\ 0 & 0 & 2*H-H^2 & 2 & 2-2*H & -2 & -2 & \dots & h^2 \\ 0 & 0 & 3*H^2-H^3 & 0 & 6*H-3*H^2 & 6 & 6-6*H & \dots & h^3 \\ 0 & 0 & 4*H^3-H^4 & 0 & 12*H^2-4*H^3 & 0 & 24*H-12*H^2 & \dots & h^4 \\ 0 & 0 & 5*H^4-H^5 & 0 & 20*H^3-5*H^4 & 0 & 60*H^2-20*H^3 & \dots & h^5 \\ 0 & 0 & 6*H^5-H^6 & 0 & 30*H^4-6*H^5 & 0 & 120*H^3-30*H^4 & \dots & h^6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{array}\right] \tag{19}$$

The polynomial is determined by the zeroth weight and evaluated at $H = h$ where it equals the Padé approximant $[n/n]$.

$$w_0 = \frac{\sum\limits^{0\le i\le n} \frac{(2*n-i)!}{(n-i)!*i!} * h^i}{\sum\limits^{0\le i\le n} (-1)^i * \frac{(2*n-i)!}{(n-i)!*i!} * h^i} = f(h); \qquad H = h \tag{20}$$

The polynomial division by smallest orders results the initial terms of the exponential series and a remainder.

$$f(h) = \sum^{0\le i\le 2*n} \frac{h^i}{i!} + \mathcal{O}\left(h^{2*n}\right) \tag{21}$$

The law of exponents applies and results the exponential function if the variable tends to zero.

$$\lim_{h\to 0} (w_0)^k = \left(\mathsf{e}^h\right)^k = \mathsf{e}^{h*k} = \mathsf{e}^x; \qquad h, k \in \mathbb{R} \tag{22}$$

Exponential Function of Single Precision

An exponential function of single precision according to IEEE 754 is determined by a rational function that equals the Padé approximant [3/3].

$$f(h) = \frac{\sum\limits^{0\le i\le 3} \frac{(6-i)!}{(3-i)!*i!} * h^i}{\sum\limits^{0\le i\le 3} (-1)^i * \frac{(6-i)!}{(3-i)!*i!} * h^i} = \frac{120 + 60*h + 12*h^2 + h^3}{120 - 60*h + 12*h^2 - h^3} \tag{23}$$

The value is computed by law of exponents with $h = 0.1$.

$$(f(0.1))^k \approx \mathsf{e}^{k*0.1} = \mathsf{e}^x \tag{24}$$

The polynomial division by smallest orders is determined in order to estimate the maximum error.

$$f(h) = \left(\sum^{0 \le i < 7} \frac{h^i}{i!} \right) + \frac{h^7}{4800} + \frac{h^8}{28800} + \mathcal{O}\left(h^9\right) \tag{25}$$

$$= \left(\sum^{0 \le i \le 7} \frac{h^i}{i!} \right) + \frac{h^7}{100800} + \frac{h^8}{28800} + \mathcal{O}\left(h^9\right) \tag{26}$$

The maximum error is estimated by the remainder compared to the single extrapolation.

$$e(h) = \left| \frac{2 * h^7}{100800} \right| + \left| \frac{2 * h^8}{28800} \right|; \qquad e(0.1) < 2.7 * 10^{-12} \tag{27}$$

The range of single precision is about $\pm 3.403 * 10^{38}$ with seven significant leading digits. The domain of the extrapolation is determined.

$$|x| = \ln\left(3.403 * 10^{38}\right) < 90 \tag{28}$$

Factor k is separated into a binary number. A maximum of nine multiplications are required for the domain of single precision and a step h.

$$90 = 900 * 0.1 < 1024 * 0.1 = 2^{10} * 0.1 \tag{29}$$

The precision of computers is finite and usually half a bit of precision is lost for each multiplication. A maximum of two multiplications is required for each binary part. Therefore a maximum of four bits of precision is lost if double precision is used for computation.

$$\log_2 \left(2 * 9 * \frac{1}{2} \right) < 4 \tag{30}$$

See [2] for more details.

References

[1] Exponential Function, Hans-Dieter Reuter, http://www.joinedpolynomials.org/exponential.pdf

[2] Joined Polynomials, Hans-Dieter Reuter, http://www.joinedpolynomials.org/jp.pdf

Listing 1: e-function of single precision in C

```
#include <math.h>
#include <stdio.h>
#include <stdlib.h>

static double wexp1n3(double const x)
{
  double const xx = x*x;
  double const A = 120.l + 12.l*xx;
  double const B = x*(60.l + xx);
  return (A+B)/(A-B);
}

double exp1(double const x)
{
  unsigned j, i; // unsigned suffices for h=0.1 and LDBL_MAX
  double wj, factor;
  // compute exponent and initial factor............................
  j = (unsigned)(fabs(x)/0.1l) + 1; // |x|/max(h)
  factor = wexp1n3(x/j); // Gewicht von x/j
  // compute power ..........................................
  wj = j&1 ? factor : 1.l; // begin with w^1 or w^0
  for(i = 2; i <= j; i <<= 1) // all exponents 2,4,8,16 <=j
  {
    factor *= factor; // w^i
    if(j&i) // if i is part of j
    {
      wj *= factor;
    }
  }
  return wj;
}

int main(int argc, char ** argv)
{
  double x, e, en;
  if(argc != 2)
  {
    fprintf(stderr, "%s x\n", argv[0]);
    exit(1);
  }
  x = atof(argv[1]);
  en = exp1(x);
  e = exp(x);
  printf("expn(%lf)=%.20lg\n", x, exp1(x));
  printf("exp (%lf)=%.20lg\n", x, exp(x));
  printf("fehler~%lg\n", (en-e)/e);
  return 0;
}
```