Polynomial Integration

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Introduction

This articles proofs that $F(x)$ is an integration of the polynomial $f(x)$ for a change v of any size.

$$f(x) = \sum^{0 \le i < n} a_i * x^i; \qquad F(x) = C + \sum^{0 \le i < n} a_i * \frac{x^{i+1}}{i+1} \tag{1}$$

The differentiation of a polynomial is discussed before its integration.

Differentiation

The variable is separated.

$$x = u + v \tag{2}$$

The separation is substituted.

$$f(u+v) = \sum^{0 \le i < n} a_i * (u+v)^i = g(u,v) \tag{3}$$

The binomial expansion is applied by v.

$$g(u,v) = \sum^{0 \le i < n} a_i * \sum^{0 \le j \le i} \binom{i}{j} * u^{i-j} * v^j \tag{4}$$

The descending faculty is defined in order to dissolve the binomial coefficient.

$$(i¡j) = \frac{i!}{(i-j)!}; \qquad (i¡0) = 1 \tag{5}$$

The sums are transposed.

$$g(u,v) = \sum^{0 \le j < n} \frac{v^j}{j!} * \sum^{j \le i < n} a_i * (i¡j) * u^{i-j} \tag{6}$$

A derivative is defined by the value of the inner sum.

$$\frac{d^j f(u)}{du^j} = \sum^{j \leq i < n} a_i * (i¡j) * u^{i-j} \tag{7}$$

The Taylor series of the polynomial is determined.

$$g(u, v) = \sum^{0 \leq j < n} \frac{v^j}{j!} * \frac{d^j f(u)}{du^j} \tag{8}$$

Integration

The integration to the next degree is defined by an offset and a constant.

$$h(u, v) = c + \sum^{0 \leq j < n} \frac{v^{j+1}}{(j+1)!} * \frac{d^j f(u)}{du^j}; \qquad \frac{d^j f(u)}{du^j} = \frac{d^{j+1} F(u)}{du^{j+1}} \tag{9}$$

The derivatives are given explicitly.

$$h(u, v) = c + \sum^{0 \leq j < n} \frac{v^{j+1}}{(j+1)!} * \sum^{j \leq i < n} a_i * (i¡j) * u^{i-j} \tag{10}$$

The sums are transposed.

$$h(u, v) = c + \sum^{0 \leq i < n} a_i * \sum^{i \leq j < n} \frac{v^{j+1}}{(j+1)!} * (i¡j) * u^{i-j} \tag{11}$$

The separation of the variable cannot be reversed due to the offset. Therefore the integration is developed at a constant location U.

$$x = u + v; \qquad u = U = \text{const}; \qquad v = x - U \tag{12}$$

The definitions are substituted and another function is determined.

$$h(U, x-U) = c + \sum^{0 \leq i < n} a_i * \sum^{i \leq j < n} \frac{(x-U)^{j+1}}{(j+1)!} * (i¡j) * U^{i-j} = F(x) \tag{13}$$

The equation is rearranged.

$$F(x) = c + \sum^{0 \leq i < n} a_i * \sum^{i \leq j < n} \frac{(i¡j)}{(j+1)!} * U^{i-j} * (x-U)^{j+1} \tag{14}$$

The binomial is expanded.

$$F(x) = c + \sum^{0 \leq i < n} a_i * \sum^{i \leq j < n} \frac{(i¡j)}{(j+1)!} * U^{i-j} * \sum^{0 \leq k \leq j+1} \binom{j+1}{k} * x^k * (-U)^{j+1-k} \tag{15}$$

The inner sums are transposed in order to group all constants.

$$F(x) = c + \sum^{0 \le i < n} a_i * \sum^{0 \le k \le i+1} x^k * \sum_{j+1 \ge k}^{0 \le j \le i} \frac{(i\text{¡}j)}{(j+1)!} * U^{i-j} * \binom{j+1}{k} * (-U)^{j+1-k} \tag{16}$$

An identity is required to rearranged the coefficients.

$$\binom{C}{B} * \binom{B}{A} = \frac{(C\text{¡}A) * ((C-A)\text{¡}(B-A))}{(B-A)! * (B\text{¡}A)} * \frac{(B\text{¡}A)}{A!} = \binom{C}{A} * \binom{C-A}{B-A} \tag{17}$$

The coefficients are rearranged such that only one factor depends on j.

$$\frac{(i\text{¡}j)}{(j+1)!} * \binom{j+1}{k} = \frac{((i+1)\text{¡}(j+1))}{(i+1) * (j+1)!} * \binom{j+1}{k} \tag{18}$$

$$= \frac{1}{i+1} * \binom{i+1}{j+1} * \binom{j+1}{k} \tag{19}$$

$$= \frac{1}{i+1} * \binom{i+1}{k} * \binom{i+1-k}{j+1-k} \tag{20}$$

These coefficients are substituted. The exponent of the constant simplifies.

$$F(x) = c + \sum^{0 \le i < n} a_i * \sum^{0 \le k \le i+1} \frac{U^{i+1-k}}{i+1} * \binom{i+1}{k} * x^k * \sum_{j+1 \ge k}^{0 \le j \le i} (-1)^{j+1-k} * \binom{i+1-k}{j+1-k} \tag{21}$$

Pascal's triangle is defined with alternating signs in order to simplify the bounds of the sums.

$$\begin{array}{ccccccccccc} & & & & 1 & & & & & = & 1 \\ & & & 1 & - & 1 & & & & = & 0 \\ & & 1 & - & 2 & + & 1 & & & = & 0 \\ & 1 & - & 3 & + & 3 & - & 1 & & = & 0 \\ 1 & - & 4 & + & 6 & - & 4 & + & 1 & = & 0 \\ & & & . & . & . & & & & = & 0 \end{array} \tag{22}$$

The zeroth line of Pascal's triangle follows under two conditions.

$$i = j; \quad i+1 = k; \qquad \frac{U^{i+1-k}}{i+1} * \binom{i+1}{k} = \frac{1}{k}; \quad (-1)^{j+1-k} * \binom{i+1-k}{j+1-k} = 1 \tag{23}$$

This case results once for each term of the integrated polynomial is noted exclusively.

$$F(x) = c + \sum^{0 \le i < n} a_i * \sum^{0 \le k < i+1} \frac{U^{i+1-k}}{i+1} * \binom{i+1}{k} * x^k * \sum_{j+1 \ge k}^{0 \le j \le i} (-1)^{j+1-k} * \binom{i+1-k}{j+1-k} \tag{24a}$$

$$+ \sum^{0 \le i < n} a_i * \frac{x^{a+1}}{i+1} \tag{24b}$$

The value of the innermost sum cancels if it maps to a full line of Pascal's triangle with alternating signs.

$$1 \le k < i+1; \qquad \sum_{j+1 \ge k}^{0 \le j \le i} (-1)^{j+1-k} * \binom{i+1-k}{j+1-k} = 0 \tag{25}$$

Therefore the bound of the middle sum simplifies to one index $k = 0$.

$$F(x) = c + \sum^{0 \le i < n} a_i * \frac{U^{i+1}}{i+1} * \sum^{0 \le j \le i} (-1)^{j+1} * \binom{i+1}{j+1} + \sum^{0 \le i < n} a_i * \frac{x^{i+1}}{i+1} \tag{26}$$

The inner sum maps to a line of Pascal's triangle with alterning signs without its zeroth element.

$$\sum^{0 \le j \le i} (-1)^{j+1} * \binom{i+1}{j+1} = -1 \tag{27}$$

The integration polynomial is determined by simple sums.

$$F(x) = c - \sum^{0 \le i < n} a_i * \frac{U^{i+1}}{i+1} + \sum^{0 \le i < n} a_i * \frac{x^{i+1}}{i+1} = C + \sum^{0 \le i < n} a_i * \frac{x^{i+1}}{i+1} \tag{28}$$

The integration polynomial simplifies if the constant equals the origin.

$$U = 0; \qquad F(x) = c + \sum^{0 \le i < n} a_i * \frac{x^{i+1}}{i+1} \tag{29}$$

See [2] for integration of polynomials of any dimension and offset.

References

[1] Polynomial Integration, Hans-Dieter Reuter, http://www.joinedpolynomials.org/integration.pdf

[2] Joined Polynomials, Hans-Dieter Reuter, http://www.joinedpolynomials.org/jp.pdf