Joined Polynomials

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Chapter 1

Introduction

I never got used to the idea that infinitisimally small quantities are neglectable ever since I have been taught so. Only years later I learned that this neglection has always been a major criticism since Leibniz and Newton developed calculus in the 17th century. I also wondered ever since school what a derivative really is since I could not think of a true equivalent in nature.

I was lucky to study Mechanical and Software Engineering in Cranfield (UK), Wuppertal (D) and Osnabrück (D). I was introduced to all the up-to-date numerical methods and a number of applications.

Having left education I never stopped thinking about those two questions. After a while I could not help considering the method of Finite Differences again and I hit on the transpositional design of polynomials eventually. From their all the methods began to unfold that are presented in this work.

I realized that differentiation and integration of polynomials can be expressed in finite terms. A derivative is thereby a mathematical pattern that is best expressed as a scalar and that many other or maybe all standard functions are applications of derivatives of polynomials. I got totally gripped and just had to give up my job to finish this work with all time available.

There are a lot of equations in this work. Therefore a few examples may get things started. For instance an equation that computes the primes below and including a bound with maxima[1].

$$\text{(\%i1)}\quad \text{length}\,(\text{ifactors}\,(24!))\,; \tag{1.1}$$

$$\text{(\%o1)}\quad 9 \tag{1.2}$$

Another straightforward equation 9.49 is an approximation of the exponential function with single precision according to IEEE 754.

$$\lim_{h\to 0}\left(\frac{120+60*h+12*h^2+h^3}{120-60*h+12*h^2-h^3}\right)^k = \mathsf{e}^{k*h}; \qquad h,k\in\mathbb{R} \tag{1.3}$$

A more complex equation is an example of the sine theorem (10.75) that results a sine exactly by a sum along the components of a Fibonacci number.

$$\sum_{0\le i\le \frac{j-1}{2}} \left\{(-1)^i * \binom{j-1-i}{i} * \sqrt{3}^{\,j-1-2*i}\right\} = 2*\sin\left(\arccos\left(\frac{1}{2}*\sqrt{3}\right)*j\right); \quad i,j\in\mathbb{N} \tag{1.4}$$

Another example shows that there are different cases for the division of polynomials as required for the approximation of the arcus tangent in Section 12.1.

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \pm ... \tag{1.5a}$$

$$\frac{1}{x^2+1} = \frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{x^6} - \frac{1}{x^8} \pm ... \tag{1.5b}$$

And there are a couple of more complicated methods and algorithms such as the integration of polynomials or differentiation of rational functions. The notation introduced in the first chapter emerged as a good toolbox to express all presented methods in compact form and is thus a good starting point.

Please report significant errors only on the lastest version of this work available at http://www.joinedpolynomials.org.

Have a nice read!

Chapter 2

Fundamentals

2.1 Expressions

The priority of an evaluation is expressed with round or curly brackets.

$$(a+b)*c = \{a+b\}*c = a*c + a*c \tag{2.1}$$

Logical 'AND' is denoted by '$*$'. Logical 'OR' is denoted by '+'. The evaluation of equalities and inequalities is expressed with round or curly brackets.

$$(a > b)*(a < b) = 0 \tag{2.2a}$$
$$(a \geq b) + (a < b) \neq 0 \tag{2.2b}$$
$$(a > b) = \{a > b\} \neq 0 \tag{2.2c}$$
$$(a < b) = \{a < b\} = 0 \tag{2.2d}$$
$$\{(a > b) \neq \{a > b\}\} = 0 \tag{2.2e}$$

A sum of expressions is denoted by a captial Sigma.

$$\sum^{0 \leq i < 3} \{i\} = 0 + 1 + 2 = 3 \tag{2.3}$$
$$\sum^{0 \leq i < 3} \{i < 2\} = \{0 < 2\} + \{1 < 2\} + \{2 < 2\} \neq 0 \tag{2.4}$$

A product of expressions is denoted by a captial Pi.

$$\prod^{0 \leq i < 3} \{i\} = 0 * 1 * 2 = 0 \tag{2.5}$$
$$\prod^{0 \leq i < 3} \{i < 2\} = \{0 < 2\} * \{1 < 2\} * \{2 < 2\} = 0 \tag{2.6}$$

2.2 Products of Constants

The faculty is defined.

$$n! = 1 * 2 * ... * (n-1) * n = \prod^{1 \leq i \leq n} \{i\} \tag{2.7}$$

A faculty contains all prime numbers up to and including n. The number of primes may be computed with maxima[1].

$$(\%i1) \quad \text{length}\,(\text{ifactors}\,(24!))\,; \tag{2.8a}$$

$$(\%o1) \quad 9 \tag{2.8b}$$

The super faculty is defined and contains at least all prime numbers up to and including n.

$$n!! = 1! * 2! * ... * (n-1)! * n! = \prod^{1 \leq j \leq n} \left\{ \prod^{1 \leq i \leq j} \{i\} \right\} \tag{2.9}$$

2.3 Quotient of Binomials

A quotient of binomials is given.

$$q = \frac{u}{v} = \frac{a+b}{c+d} \tag{2.10}$$

The quotient is separable into four different expressions. These expressions consist of further quotients of binomials. Repeated separation may give a convergent series of the original quotient. Repeated separation also gives the division of a rational function, see section 8.2.

The quotient is separated by a and c.

$$\frac{a+b}{c+d} = \frac{a}{c} + \frac{b}{c+d} - \frac{a}{c} * \frac{d}{c+d} \tag{2.11}$$

The quotient is separated by b and c.

$$\frac{b+a}{c+d} = \frac{b}{c} + \frac{a}{c+d} - \frac{b}{c} * \frac{d}{c+d} \tag{2.12}$$

The quotient is separated by a and d.

$$\frac{a+b}{d+c} = \frac{a}{d} + \frac{b}{d+c} - \frac{a}{d} * \frac{c}{d+c} \tag{2.13}$$

The quotient is separated by b and d.

$$\frac{b+a}{d+c} = \frac{b}{d} + \frac{a}{d+c} - \frac{b}{d} * \frac{c}{d+c} \tag{2.14}$$

2.4 Pascal Patterns

Patterns of faculties are refered to as Pascal Patterns.

2.4.1 Pascal's Triangle

Each line of Pascal's triangle gives the coefficients for the expansion of a power of a binomial, see also [2, p321].

$$a^3 = (b+c)^3 = 1*b^3*c^0 + 3*b^2*c^1 + 3*b^1*c^2 + 1*b^0*c^3 \tag{2.15}$$

The sum of elements of each line is given with positive and alternating signs.

Exponent												$\sum +$	$\sum \pm$
0						1						2^0	1
1					1		± 1					2^1	0
2				1		± 2		1				2^2	0
3			1		± 3		3		± 1			2^3	0
4		1		± 4		6		± 4		1		2^4	0
5	1		± 5		10		± 10		5		± 1	2^5	0
.				.		.		.				.	.

(2.16)

Lines begin and end with 1. An Element inbetween is the sum of two elements of the preceding line.

$$e[j+1][i+1] = e[j][i] + e[j][i+1]\,; \qquad 1 \le i \le j \tag{2.17}$$

2.4.2 Rectangular Pascal Pattern

The rectangular Pascal pattern is given.

$$\binom{i+j}{i} = \binom{i+j}{j} \tag{2.18}$$

i	0	1	2	3	4	5	6	7	8	9	
j	1	1	1	1	1	1	1	1	1	1	...
1	1	2	3	4	5	6	7	8	9	10	...
2	1	3	6	10	15	21	28	36	45	55	...
3	1	4	10	20	35	56	84	120	165	220	...
4	1	5	15	35	70	126	210	330	495	715	...

An element is the sum of its upper and left neighbour.

$$e[j][i] = e[j-1][i] + e[j][i-1]\,; \qquad i > 0;\; j > 0 \tag{2.19}$$

Therefore an element is the sum of all elements of the line above up to and including the element of the same column.

$$e[j+1][k] = \sum^{0 \le i \le k} \{e[j][i]\} \tag{2.20}$$

Sums of the first n natural numbers are given by the zero-based second line.

$$\frac{1}{2} * n * (n+1) = \binom{i+1}{2} = \sum^{1 \le i \le n} \{i\} \tag{2.21}$$

2.4.3 Exponential Pascal Pattern

The exponential Pascal pattern is the square of the e-series, see section 9.2. A diagonal describes a line of Pascal's triangle divided by the faculty of the index of the diagonal.

$$\frac{1}{i! * j!} \tag{2.22}$$

i		0	1	2	3	4	5	...
		1/0!	1/1!	1/2!	1/3!	1/4!	1/5!	...
j	1/0!	1/1	1/1	1/2	1/6	1/24	1/120	...
1	1/1!	1/1	1/1	1/2	1/6	1/24	...	...
2	1/2!	1/2	1/2	1/4	1/12	...	...	...
3	1/3!	1/6	1/6	1/12	...	...	...	...
4	1/4!	1/24	1/24	...	...	...	...	...
5	1/5!	1/120	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...

2.4.4 Parallelogram Pascal Pattern

The parallelogram Pascal pattern is given and maps to the sine theorem (10.75). Each line of absolute values sums to a Fibonacci number. Each line of alternating sign maps to a sine.

$$\binom{j-i}{i} \tag{2.23}$$

i	0	1	2	$\sum +$	$\sum \pm$
j	1			1	1
1	1			1	1
2	1	± 1		1	0
3	1	± 2		3	-1
4	1	± 3	1	5	-1

i	0	1	2	3	4	$\sum +$	$\sum \pm$
j	1	± 4	3			8	0
6	1	$\pm\underline{5}$	6	± 1		13	1
7	1	± 6	$\underline{10}$	± 4		21	1
8	1	± 7	$\underline{15}$	± 10	1	34	0
9	1	± 8	21	± 20	5	55	-1

An element is given.

$$c[j+2][i+1] = c[j+1][i+1] + c[j][i] \tag{2.24}$$

2.5 Real Number

The set of real numbers is denoted by $\mathbb{R}$, see [2, p360].

The set of all real numbers between and including a and b is defined.

$$r = \mathbb{R}\mathrm{AB}\,(a; b) \tag{2.25}$$

An arbitrary real number or variable is expressed by a latin character such as a or x.

Real numbers are frequently combined to vectors, matrices or tensors and combinations of those. See [2, p271,matrix], [3, p209] or others for details.

2.6 Method

A method is an operator that maps a number of arguments to exactly one value.

$$y = f(...) \tag{2.26}$$

A function is a method that maps exactly one argument to one value.

$$y = g(x) \tag{2.27}$$

2.7 Array

An array is a finite ordered list of elements such as constants, equations or other arrays. $\mathbb{A}$ is the set of all arrays. Elements of an array are separated with ';'. The value of an array is given in angular brackets.

$$a = \langle -2; 3; 1\rangle\,; \qquad b = \langle 1; 2; 4\rangle\,; \qquad c = \langle a; b; 1\rangle \tag{2.28}$$

The method 'size' is defined and gives the number of elements of an array.

$$\text{size}\,(a) = \text{size}\,(\langle -2; 3; 1\rangle) = 3 \tag{2.29}$$

The operator '[]' is defined and gives a zero-based element of an array.

$$a[0] = -2 \qquad a[1] = 3 \qquad a[2] = 1 \tag{2.30}$$

2.8 Vector

A vector is an array of real numbers. $\mathbb{V}$ is the set of all vectors.

$$a \in \mathbb{V} = \sum^{0 \le i < \text{size}(a)} \langle a[i] \in \mathbb{R}\rangle \tag{2.31}$$

The method '$\mathcal{P}$' is defined and gives the check product of a vector.

$$\mathcal{P}(a) = \prod^{0 \le i < N} \{a[i]\} = -2 * 3 * 1 = -6; \qquad N = \text{size}\,(a) \tag{2.32}$$

The method '$\mathcal{S}$' is defined and gives the horizontal check sum of a vector.

$$\mathcal{S}(a) = \sum^{0 \le i < N} \{a[i]\} = -2 + 3 + 1 = 2; \qquad N = \text{size}\,(a) \tag{2.33}$$

The method 'chkpow' is defined and gives a check power of a vector.

$$\text{chkpow}(a; b) = \prod^{0 \le i < N} \left\{ a[i]^{b[i]} \right\} = -2^1 * 3^2 * 1^4 = -18; \qquad N = \text{size}\,(a) = \text{size}\,(b) \tag{2.34}$$

The method 'max∈' is defined. It gives the element with the largest value of all elements.

$$\max\in(\langle 3\rangle) = 3; \qquad \max\in(\langle 0;2\rangle) = 2; \qquad \max\in(\langle 1;0;2\rangle) = 2 \tag{2.35}$$

The method iAbsMax(a) is defined. It gives the index of that element of vector a with the largest absolute value of all elements.

$$\text{iAbsMax}\left(\left\langle -\frac{1}{2}; \frac{4}{3}; -\frac{11}{6}\right\rangle\right) = 2 \tag{2.36}$$

Comparison operators are defined on the elements of a vector. A vector can be neither equal, less nor greater than another vector.

$$\langle 1;0\rangle \neq \langle 2;1\rangle\,; \qquad \langle 1;0\rangle < \langle 2;1\rangle\,; \qquad \langle 1;0\rangle \ngtr \langle 2;1\rangle \tag{2.37}$$

$$\langle 3;1\rangle \neq \langle 2;2\rangle\,; \qquad \langle 3;1\rangle \nless \langle 2;2\rangle\,; \qquad \langle 3;1\rangle \ngtr \langle 2;2\rangle \tag{2.38}$$

The basic arithmetic operations are defined on the elements of a vector.

$$\langle 1;0\rangle + \langle 2;1\rangle = \langle 3;1\rangle\,; \qquad \langle 3;1\rangle - \langle 2;0\rangle = \langle 1;1\rangle\,; \qquad \langle 3;1\rangle - \langle 2;2\rangle = \langle 1;-1\rangle \tag{2.39}$$

$$\langle 1;0\rangle * \langle 2;1\rangle = \langle 2;0\rangle\,; \qquad \langle 6;3\rangle \,/\, \langle 3;1\rangle = \langle 2;3\rangle\,; \qquad \langle 6;3\rangle \,/\, \langle 4;1\rangle = \langle 3/2;3\rangle \tag{2.40}$$

The method sdspanAB$(a;b)$ is defined and gives the span of two vectors by equal sums of indexes.

$$\langle a;b\rangle \in \mathbb{V};\ N = \text{size}(a) + \text{size}(b) \tag{2.41a}$$

$$\text{sdspanAB}(a;b) = \sum^{0\le k<N} \left\langle \sum^{0\le j<k} \left\{ \sum_{i+j=k}^{0\le i<k} \{a[i]*b[j]\} \right\} \right\rangle \tag{2.41b}$$

$$\text{sdspanAB}(a;b) = \begin{pmatrix} a[0]*b[0]\,; \\ a[0]*b[1] + a[1]*b[1]\,; \\ a[0]*b[2] + a[1]*b[1] + a[2]*b[0]\,; \\ \ldots \end{pmatrix} \tag{2.41c}$$

2.9 Finite Number Series in Terms of Vectors

A series of arbitrarily terms or numbers $a[i]$ is convergent if the series tends to a constant for arbitrarily many terms.

$$a \in \mathbb{V};\ N = \text{size}(a) \tag{2.42}$$

$$\text{if}\left(\lim_{N\to\infty}\left(\sum^{0\le i<N}\{a[i]\}\right) \to \text{const}\right)\text{then}\,(a \text{ converges}) \tag{2.43}$$

A number series is prime if its elements have no common factors but ± 1.

$$a;b \in \mathbb{V};\ N = \text{size}(a) = \text{size}(b) \tag{2.44}$$

$$\text{if}\left((\text{abs}(b) \neq 1) * \prod^{0\le i<N}\{a[i] = b * c[i]\}\right)\text{then}\,(a \text{ is not prime}) \tag{2.45}$$

It is assumed that the Euler number series or $\mathtt{e}$-series tends to the largest constant of all prime series, see section 9.2.

$$\lim_{N\to\infty}\left(\pm\sum^{0\le i<N}\left\{\frac{1}{i!}\right\}\right)\to\pm\mathtt{e} \tag{2.46}$$

The range of any other series is within $\pm\mathtt{e}$ if all terms of that series are less than or equal to the corresponding term of the $\mathtt{e}$-series.

$$\text{if}\left(\prod^{0\le i<N}\left\{a[i]\le\frac{1}{i!}\right\}\right)\text{then}\left(-\mathtt{e}<\sum^{0\le i<N}\{a[i]\}<\mathtt{e}\right);\qquad a\in\mathbb{V};\ N=\text{size}\,(a) \tag{2.47}$$

A number series is strictly stable if the absolute value of each term is greater than double the absolute value of the successive term.

$$\text{if}\left(\sum^{0\le i<\text{size}(a)-1}\{\text{abs}(a[i])>2*\text{abs}(a[i+1])\}\right)\text{then}\,(a\text{ is strictly stable});\qquad a\in\mathbb{V} \tag{2.48a}$$

Two is separated repeatedly. Only seven non-zero terms determine 99% of the value of a strictly stable series with terms of equal sign.

$$2=\frac{2}{2-1}=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+A=1.984375+A \tag{2.48b}$$

The number of required terms i for a remainder r is determined.

$$\left[\frac{1}{2}*r*100\right]=\%;\qquad r>2*\frac{1}{2^i}=\frac{1}{2^{i-1}};\qquad \log\left[\frac{1}{2}\right](r)=\lfloor i+1\rfloor \tag{2.48c}$$

Two is the largest constant that a prime strictly stable series tends to by this definition.

$$\lim_{N\to\infty}\left(\pm\sum^{0\le i<N}\left\{\frac{1}{2^i}\right\}\right)\to\pm 2;\qquad \sum^{0\le i<N}\left\{2^i\right\}<2^N \tag{2.48d}$$

A stable finite number series is self-accurate of zeroth order. A stable finite number series is self-accurate of first order if the span by equal sums of indexes (2.41b) is stable. A stable finite number series is self-accurate of order O if the O-fold span by equal sums of indexes is stable.

An exponential series of four terms is given. It is self-accurate of second order.

$$a=\left(\frac{1}{5^0*0!};\frac{1}{5^1*1!};\frac{1}{5^2*2!};\frac{1}{5^3*3!}\right)=\frac{1}{750}*(750;150;15;1) \tag{2.49a}$$

$$b=\text{sdspanAB}\,(a;a)=\frac{1}{375}*(375;150;30;4;...) \tag{2.49b}$$

$$c=\text{sdspanAB}\,(b;b)=\frac{1}{5000}*(5000;3000;900;180;27;...) \tag{2.49c}$$

$$d=\text{sdspanAB}\,(c;c)=\frac{1}{1875}*(1875;1500;600;160;32;...) \tag{2.49d}$$

$$1875<1500+600+160+32+...=2292+... \tag{2.49e}$$

2.10 Tuple

A tuple is a vector of natural numbers and denoted by a greek letter. The set of all tuples is $\mathbb{T}$.

$$\alpha \in \mathbb{T} = \sum^{0 \le i < \text{size}(\alpha)} \langle \alpha[i] \in \mathbb{N} \rangle \,; \qquad \begin{matrix} \alpha = \langle 3 \rangle \\ \beta = \langle 0; 2 \rangle \\ \gamma = \langle 1; 0; 2 \rangle \end{matrix} \tag{2.50}$$

The set of all tuples with n elements $\mathbb{T}\text{n}(n)$ is defined.

$$\text{if}\,(\alpha \in \mathbb{T}\text{n}(n))\,\text{then}\,((\alpha \in \mathbb{T}) * (\text{size}\,(\alpha) = n)) \tag{2.51}$$

A tuple of identical elements is denoted by a bold natural number.

$$\mathbf{0} = \langle 0; 0; ...; 0 \rangle \qquad \mathbf{1} = \langle 1; 1; ...; 1 \rangle \tag{2.52}$$

A difference of two tuples does not exist if any element of the minuend is less than the corresponding element of the subtrahend.

$$\langle 3; 1 \rangle - \langle 2; 2 \rangle = \nexists \tag{2.53}$$

A quotient of two tuples does not exist if any element of the dividend is not a natural multiple of the corresponding element of the divisor.

$$\langle 6; 3 \rangle \,/\, \langle 4; 1 \rangle = \nexists \tag{2.54}$$

The faculty is defined on the elements of a tuple. The check product of the faculty is denoted by an asterisk at the subscript position.

$$(\langle 1; 3; 0 \rangle)! = \langle 1!; 3!; 0! \rangle = \langle 1; 6; 1 \rangle \qquad (\langle 1; 3; 0 \rangle)!_* = 6 \tag{2.55}$$

The descending faculty and its check product are defined.

$$(\alpha¡\beta) = \frac{\alpha!}{(\alpha - \beta)!} \qquad (\alpha¡\beta)_* = \mathcal{P}((\alpha¡\beta)) \tag{2.56}$$

$$(\langle 1; 3; 0 \rangle \,¡\, \langle 1; 2; 0 \rangle) = \left\langle \frac{1!}{0!}; \frac{3!}{1!}; \frac{0!}{0!} \right\rangle = \langle 1; 6; 1 \rangle \qquad (\langle 1; 3; 0 \rangle \,¡\, \langle 1; 2; 0 \rangle)_* = 6 \tag{2.57}$$

The descending faculty of a constant and zero equals One.

$$(a¡0) = 1 \tag{2.58}$$

The binomial coefficient is defined on the elements of a tuple. The check product of the binomial coefficient is denoted by an asterisk.

$$\binom{\langle 1; 3; 0 \rangle}{\langle 1; 2; 0 \rangle} = \left\langle \binom{1}{1}; \binom{3}{2}; \binom{0}{0} \right\rangle = \langle 1; 3; 1 \rangle \,; \qquad \binom{\langle 1; 3; 0 \rangle}{\langle 1; 2; 0 \rangle}_* = \binom{1}{1} * \binom{3}{2} * \binom{0}{0} = 3 \tag{2.59}$$

A specific equalty is given which is required for polynomial integration (5.10b).

$$\binom{\gamma}{\beta} * \binom{\beta}{\alpha} = \binom{\gamma}{\alpha} * \binom{\gamma - \alpha}{\beta - \alpha}; \qquad \alpha \le \beta \le \gamma \tag{2.60a}$$

The binomial coefficients are given explicitly.

$$\binom{\gamma}{\beta} * \binom{\beta}{\alpha} = \frac{(\gamma ¡ \beta)_*}{\beta!_*} * \frac{(\beta ¡ \alpha)_*}{\alpha!_*} \tag{2.60b}$$

Numerator and denominator of the first quotient are factorized to give descending faculties of α.

$$\binom{\gamma}{\beta} * \binom{\beta}{\alpha} = \frac{(\gamma ¡ \alpha)_* * ((\gamma - \alpha) ¡ (\beta - \alpha))_*}{(\beta - \alpha)!_* * (\beta ¡ \alpha)} * \frac{(\beta ¡ \alpha)_*}{\alpha!_*} \tag{2.60c}$$

Some factors are rearranged.

$$\binom{\gamma}{\beta} * \binom{\beta}{\alpha} = \frac{(\gamma ¡ \alpha)_*}{\alpha!_*} * \frac{((\gamma - \alpha) ¡ (\beta - \alpha))_* * (\beta ¡ \alpha)_*}{(\beta - \alpha)!_* * (\beta ¡ \alpha)} \tag{2.60d}$$

Two factors cancel and equalty 2.60a is obtained.

$$\binom{\gamma}{\beta} * \binom{\beta}{\alpha} = \frac{(\gamma ¡ \alpha)_*}{\alpha!_*} * \frac{((\gamma - \alpha) ¡ (\beta - \alpha))_*}{(\beta - \alpha)!_*} = \binom{\gamma}{\alpha} * \binom{\gamma - \alpha}{\beta - \alpha} \tag{2.60e}$$

2.11 Array of Tuples

An array of tuples is complete if it contains all elements that satisfy specific conditions.

The method $\mathbb{T}\mathrm{ns}\,(n; s)$ is defined and gives the complete array of tuples with n elements and a check sum of s. The order of elements is significant.

$$\mathbb{T}\mathrm{ns}\,(2; 2) = \langle \langle 0; 2 \rangle ; \langle 1; 1 \rangle ; \langle 2; 0 \rangle \rangle \tag{2.61}$$

The method $\mathbb{T}\mathrm{nss}\,(n; s)$ is defined and gives the complete array of tuples with n elements and a check sum of less or equal s. The order of elements is significant.

$$\mathbb{T}\mathrm{nss}\,(2; 2) = (\langle 0; 0 \rangle ; \langle 0; 1 \rangle ; \langle 1; 0 \rangle ; \langle 0; 2 \rangle ; \langle 1; 1 \rangle ; \langle 2; 0 \rangle) \tag{2.62}$$

The degree of an array of tuples is defined as the largest check sum of all contained elements.

$$\mathcal{G}\left(\mathbb{T}\mathrm{nss}\,(n; s)\right) = s \tag{2.63}$$

$$\mathcal{G}\left(\langle \langle 0; 1 \rangle ; \langle 1; 1 \rangle ; \langle 1; 2 \rangle \rangle\right) = 3 \tag{2.64}$$

The vertical check sum $\mathcal{S}\mathbb{T}\,(a)$ of an array of tuples is defined. It is the tupel of all sums along the elements of the contained tupels.

$$\mathcal{S}\mathbb{T}\,(a) = \sum^{0 \le i < n} \left\langle \sum^{0 \le j < N} \{a[j][i]\} \right\rangle ; \qquad N = \mathrm{size}\,(a) \tag{2.65}$$

$$\mathcal{S}\mathbb{T}\left(\langle \langle 0; 1 \rangle ; \langle 1; 1 \rangle ; \langle 1; 2 \rangle \rangle\right) = \langle 0 + 1 + 1; 1 + 1 + 2 \rangle = \langle 2; 4 \rangle \tag{2.66}$$

The number of unique tuples with n elements and a given check sum equals the value of a binomial coefficient.

$$\mathrm{size}\left(\mathbb{T}\mathrm{ns}\,(n; s)\right) = \binom{n + s - 1}{s} \tag{2.67}$$

$$\mathrm{size}\left(\mathbb{T}\mathrm{ns}\,(2; 3)\right) = \mathrm{size}\left(\langle 0; 3 \rangle ; \langle 1; 2 \rangle ; \langle 2; 1 \rangle ; \langle 3; 0 \rangle\right) = \binom{2 + 3 - 1}{3} = 4 \tag{2.68}$$

The number of unique tuples with n elements and a check sum of less or equal a given constant equals a binomial coefficient.

$$\text{size}\left(\mathbb{T}\text{nss}\left(n; s\right)\right) = \binom{n+s}{s} = c \tag{2.69a}$$

$$\text{size}\left(\mathbb{T}\text{nss}\left(2; 2\right)\right) = \text{size}\left(\langle 0; 0\rangle ; \langle 0; 1\rangle ; \langle 1; 0\rangle ; \langle 0; 2\rangle ; \langle 1; 1\rangle ; \langle 2; 0\rangle\right) = \binom{2+2}{2} = 6 \tag{2.69b}$$

The method $\text{gSnt}\left(n; c\right)$ is defined implicitly as an inverse operation of (2.69a). It gives the largest possible degree of a complete array with less than c tuples of a check sum less or equal s.

$$\binom{n + \text{gSnt}\left(n; c\right)}{s} < c; \qquad \begin{array}{l} \text{gSnt}\left(2; 5\right) = 1 \\ \text{gSnt}\left(2; 6\right) = 1 \\ \text{gSnt}\left(2; 7\right) = 2 \end{array} \tag{2.69c}$$

Addition of arrays of tuples results another array of tuples.

$$a + b = c; \qquad a; b; c \in \mathbb{A} \tag{2.70}$$

$$\langle \alpha \rangle + \langle \beta; \gamma \rangle = \langle \alpha; \beta; \gamma \rangle \tag{2.71}$$

2.12 Array of Arrays of Tuples

An array may contain arbitrary arrays of tuples. An array of arrays with equal numbers of tuples is called a matrix.

$$\begin{array}{cc} \text{Arbitrary} & \text{Matrix} \\ \left(\begin{array}{l} \langle \alpha; \beta \rangle ; \\ \langle \gamma; \delta; \epsilon \rangle ; \\ \langle \zeta; \eta \rangle \end{array}\right) & \left(\begin{array}{l} \langle \alpha; \beta; \mathbf{0} \rangle ; \\ \langle \gamma; \delta; \epsilon \rangle ; \\ \langle \zeta; \eta; \mathbf{0} \rangle \end{array}\right) \end{array} \tag{2.72}$$

The method $\mathbb{CT}\text{n}\left(T; n\right)$ is defined. It gives the matrix of all combinations of n tuples of array T. The order of elements is significant.

$$\alpha = \langle 0; 2\rangle ; \quad \beta = \langle 1; 1\rangle ; \quad \gamma = \langle 2; 0\rangle \tag{2.73a}$$

$$T = \mathbb{T}\text{ns}\left(2; 2\right) = \langle \alpha; \beta; \gamma \rangle \tag{2.73b}$$

$$C = \mathbb{CT}\text{n}\left(T; 2\right) = \langle \langle \alpha; \beta \rangle ; \langle \alpha; \gamma \rangle ; \langle \beta; \gamma \rangle \rangle \tag{2.73c}$$

$$C[1] = \langle \alpha; \gamma \rangle ; \quad C[1][0] = \alpha; \quad C[1][0][1] = 2 \tag{2.73d}$$

The method $\text{long}\alpha n(\alpha; n)$ is defined. It gives an array of all arrays with n tuples and a vertical check sum equal to α. The order of elements is significant. The method is required by the differentiation of a product of polynomials, page 31.

$$\text{long}\alpha n(\langle 1; 2\rangle ; 2) = \left(\begin{array}{c} \langle \langle 0; 0\rangle ; \langle 1; 2\rangle \rangle ; \quad \langle \langle 0; 1\rangle ; \langle 1; 1\rangle \rangle ; \quad \langle \langle 1; 0\rangle ; \langle 0; 2\rangle \rangle ; \\ \langle \langle 0; 2\rangle ; \langle 1; 0\rangle \rangle ; \quad \langle \langle 1; 1\rangle ; \langle 0; 1\rangle \rangle ; \quad \langle \langle 1; 2\rangle ; \langle 0; 0\rangle \rangle \end{array} \right) \tag{2.74}$$

The method $\text{spanccAB}\left(a; b\right)$ is defined. It gives the rectangular span of arrays a and b by concatenation as a matrix.

$$\text{spanccAB}\left(a; b\right) = \sum^{0 \le j < \text{size}(b)} \left\langle \sum^{0 \le i < \text{size}(a)} \langle b[j] ; a[i] \rangle \right\rangle \tag{2.75}$$

The method spanccAS $(a; s)$ is defined in order to define method spancc$\mathbb{T}$e $(a; e)$ next. It gives the rectangular span of array a and matrix s by addition as a matrix. Therefore each element of array a is added as an array.

$$\text{spanccAS}\,(a; s) = \sum^{0\le j<\text{size}(s)} \left\langle \sum^{0\le i<\text{size}(a)} \langle s[j] + \langle a[i]\rangle\rangle \right\rangle \tag{2.76}$$

The method spancc$\mathbb{T}$e $(a; e)$ is defined. It gives the e-fold rectangular span of array a by concatenation as a matrix. This method is required for the multiplication of polynomials, page 25, 31, 33.

$$t[0] = \langle a\rangle\,; \qquad \sum^{1\le i<e} \langle t[i] = \text{spanccAS}\,(a; t[i-1])\rangle\,; \qquad \text{spancc}\mathbb{T}\text{e}\,(a; e) = t[e-1] \tag{2.77}$$

The three-fold span of an array of two tuples is given.

$$\begin{array}{c|cc} & \alpha & \beta \\ \hline \alpha & \langle\alpha;\alpha\rangle & \langle\alpha;\beta\rangle \\ \beta & \langle\beta;\alpha\rangle & \langle\beta;\beta\rangle \end{array}\;; \qquad \begin{array}{c|cc} & \alpha & \beta \\ \hline \langle\alpha;\alpha\rangle & \langle\alpha;\alpha;\alpha\rangle & \langle\alpha;\alpha;\beta\rangle \\ \langle\alpha;\beta\rangle & \langle\alpha;\beta;\alpha\rangle & \langle\alpha;\beta;\beta\rangle \\ \langle\beta;\alpha\rangle & \langle\beta;\alpha;\alpha\rangle & \langle\beta;\alpha;\beta\rangle \\ \langle\beta;\beta\rangle & \langle\beta;\beta;\alpha\rangle & \langle\beta;\beta;\beta\rangle \end{array} \tag{2.78a}$$

$$\text{spancc}\mathbb{T}\text{e}\,(\langle\alpha;\beta\rangle\,; 3) = \left\langle \begin{array}{r} \langle\alpha;\alpha;\alpha\rangle\,; \langle\alpha;\alpha;\beta\rangle\,; \langle\alpha;\beta;\alpha\rangle\,; \langle\alpha;\beta;\beta\rangle\,; \\ \langle\beta;\alpha;\alpha\rangle\,; \langle\beta;\alpha;\beta\rangle\,; \langle\beta;\beta;\alpha\rangle\,; \langle\beta;\beta;\beta\rangle \end{array} \right\rangle \tag{2.78b}$$

The two-fold span of an array of three tuples is given.

$$\text{spancc}\mathbb{T}\text{e}\,(\langle\alpha;\beta;\gamma\rangle\,; 2) = \left\langle \begin{array}{r} \langle\alpha;\alpha\rangle\,; \langle\alpha;\beta\rangle\,; \langle\alpha;\gamma\rangle\,; \langle\beta;\alpha\rangle\,; \\ \langle\beta;\beta\rangle\,; \langle\beta;\gamma\rangle\,; \langle\gamma;\alpha\rangle\,; \langle\gamma;\beta\rangle\,; \langle\gamma;\gamma\rangle \end{array} \right\rangle \tag{2.79}$$

2.13 Array of Arrays of Arrays of Tupels

The method long$\mathbb{TT}(T)$ is defined. It gives the division of the array of arrays T by the vertical check sum. The method is required for the multiplication of polynomials, page 25, 31, 33. The division of a matrix is given. A simple definition without additional methods is unknown.

$$\alpha = \langle 0;0\rangle\,; \quad \beta = \langle 0;1\rangle\,; \quad \gamma = \langle 1;0\rangle \tag{2.80a}$$

$$A = \mathbb{T}\text{nss}\,(2;1) = \langle\alpha;\beta;\gamma\rangle \tag{2.80b}$$

$$B = \text{spancc}\mathbb{T}\text{e}\,(A;2) = (\langle\alpha;\alpha\rangle\,; \langle\alpha;\beta\rangle\,; \langle\alpha;\gamma\rangle\,; \langle\beta;\alpha\rangle\,; \langle\beta;\beta\rangle\,; \langle\beta;\gamma\rangle\,; \langle\gamma;\alpha\rangle\,; \langle\gamma;\beta\rangle\,; \langle\gamma;\gamma\rangle) \tag{2.80c}$$

$$C = \text{long}\mathbb{TT}(B) = \left(\begin{array}{l} (\langle\alpha;\alpha\rangle)\,; \\ (\langle\alpha;\beta\rangle\,; \langle\beta;\alpha\rangle)\,; \\ (\langle\alpha;\gamma\rangle\,; \langle\gamma;\alpha\rangle)\,; \\ (\langle\beta;\beta\rangle)\,; \\ (\langle\beta;\gamma\rangle\,; \langle\gamma;\beta\rangle)\,; \\ (\langle\gamma;\gamma\rangle) \end{array} \right) \tag{2.80d}$$

2.14 Distance, Position and Point

A vector that describes a distance in a space is marked with an arrow $\vec{a}$, see also [2, vector]. Such a segment is called a position vector or position if it is located at the original position $\vec{\mathbf{0}}$.

A point is defined as the pair of a position and the value of a function of that position.

$$A = \big(\, \langle \vec{x}; f(\vec{x}) \rangle \,\big) \tag{2.81}$$

The power [2, exponent] of a distance is defined as the product of powers of the contained elements. Here only exponents of natural numbers or tuples are required. Powers of distances make the basis of polynomials.

$$\vec{x}^{\alpha} = \prod^{0 \le i < N} \left\{ \vec{x}[i]^{\alpha[i]} \right\}; \qquad N = \mathrm{size}\,(\alpha) = \mathrm{size}\,(\vec{x}) \tag{2.82}$$

The power of a distance is a real number and not another distance. Therefore the exponent is to be stated for a power of a distance.

$$\overrightarrow{\langle 2; 3 \rangle}^{\langle 1;1 \rangle} = 2^1 * 3^1 = 6 \neq \langle 2; 3 \rangle \tag{2.83}$$

The factorization of a natural number into primes also results a power of a distance.

$$25872 = 2^4 * 3^1 * 7^2 * 11^1 = \overrightarrow{2; 3; 7; 11}^{4;1;2;1} \tag{2.84}$$

The binomial expansion [3, p70] also applies to powers of a sum of two distances. A power of a sum of two distances is given.

$$N = \mathrm{size}\,(\alpha) = \mathrm{size}\,(\vec{u}) = \mathrm{size}\,(\vec{v}) \tag{2.85a}$$

$$(\vec{u} + \vec{v})^{\alpha} = \prod^{0 \le i < N} \left\{ (\vec{u}[i] + \vec{v}[i])^{\alpha[i]} \right\} \tag{2.85b}$$

The expansion is applied in terms of the elements.

$$(\vec{u} + \vec{v})^{\alpha} = \prod^{0 \le i < N} \left\{ \sum^{0 \le j < \alpha[i]} \left\{ \binom{\alpha[i]}{j} * \vec{u}[i]^{\alpha[i] - j} * \vec{v}[i]^{j} \right\} \right\} \tag{2.85c}$$

An analytical method of combining the elements to powers of distances is unknown. However, it may be shown that the expansion equals the given rule in each case.

$$(\vec{u} + \vec{v})^{\alpha} = \sum_{\beta \le \alpha}^{\beta \in T} \left\{ \binom{\alpha}{\beta}_{\!*} * \vec{u}^{\alpha - \beta} * \vec{v}^{\beta} \right\}; \qquad T = \mathbb{T}\mathrm{nss}\,(N;\, \max\!\in(\alpha)) \tag{2.85d}$$

An example is given.

$$\begin{aligned}(\vec{u} + \vec{v})^{\langle 2;3 \rangle} &= \sum_{\beta \le \langle 2;3 \rangle}^{\beta \in \mathbb{T}\mathrm{ns}(\mathrm{size}(\langle 2;3 \rangle);\, \max\in(\langle 2;3 \rangle))} \left\{ \binom{\langle 2; 3 \rangle}{\beta} * \vec{u}^{\alpha - \beta} * \vec{v}^{\beta} \right\} \\ &= \vec{u}^{\langle 2;3 \rangle} * \vec{v}^{\langle 0;0 \rangle} + 3 * \vec{u}^{\langle 2;2 \rangle} * \vec{v}^{\langle 0;1 \rangle} + 2 * \vec{u}^{\langle 1;3 \rangle} * \vec{v}^{\langle 1;0 \rangle} + 3 * \vec{u}^{\langle 2;1 \rangle} * \vec{v}^{\langle 0;2 \rangle} \\ &\quad + 6 * \vec{u}^{\langle 1;2 \rangle} * \vec{v}^{\langle 1;1 \rangle} + \vec{u}^{\langle 0;3 \rangle} * \vec{v}^{\langle 2;0 \rangle} + \vec{u}^{\langle 2;0 \rangle} * \vec{v}^{\langle 0;3 \rangle} + 6 * \vec{u}^{\langle 1;1 \rangle} * \vec{v}^{\langle 1;2 \rangle} \\ &\quad + 3 * \vec{u}^{\langle 0;2 \rangle} * \vec{v}^{\langle 2;1 \rangle} + 2 * \vec{u}^{\langle 1;0 \rangle} * \vec{v}^{\langle 1;3 \rangle} + 3 * \vec{u}^{\langle 0;1 \rangle} * \vec{v}^{\langle 2;2 \rangle} + \vec{u}^{\langle 0;0 \rangle} * \vec{v}^{\langle 2;3 \rangle}\end{aligned} \tag{2.86}$$

2.15 Vandermonde Matrices

A Vandermonde matrix is a regular matrix of powers of one-dimensional positions.

$$V = \sum^{0\leq j<n} \left\langle \sum^{0\leq i<n} \left\langle \vec{x}[i]^j \right\rangle \right\rangle = \begin{bmatrix} \vec{x}[0]^0 & \vec{x}[0]^1 & \vec{x}[0]^2 & ... & \vec{x}[0]^{n-1} \\ \vec{x}[1]^0 & \vec{x}[1]^1 & \vec{x}[1]^2 & ... & \vec{x}[1]^{n-1} \\ \vec{x}[2]^0 & \vec{x}[2]^1 & \vec{x}[2]^2 & ... & \vec{x}[2]^{n-1} \\ ... & ... & ... & ... & ... \\ \vec{x}[n-1]^0 & \vec{x}[n-1]^1 & \vec{x}[n-1]^2 & ... & \vec{x}[n-1]^{n-1} \end{bmatrix} \tag{2.87}$$

A transposed Vandermonde matrix follows accordingly.

$$VT = \sum^{0\leq j<n} \left\langle \sum^{0\leq i<n} \left\langle \vec{x}[j]^i \right\rangle \right\rangle = \begin{bmatrix} \vec{x}[0]^0 & \vec{x}[1]^0 & \vec{x}[2]^0 & ... & \vec{x}[n-1]^0 \\ \vec{x}[0]^1 & \vec{x}[1]^1 & \vec{x}[2]^1 & ... & \vec{x}[n-1]^1 \\ \vec{x}[0]^2 & \vec{x}[1]^2 & \vec{x}[2]^2 & ... & \vec{x}[n-1]^2 \\ ... & ... & ... & ... & ... \\ \vec{x}[0]^{n-1} & \vec{x}[1]^{n-1} & \vec{x}[2]^{n-1} & ... & \vec{x}[n-1]^{n-1} \end{bmatrix} \tag{2.88}$$

The determinant of a Vandermonde matrix is the product of all possible differences.

$$\det(V) = \det(VT) = \prod^{1\leq j<n} \left\{ \prod^{0\leq i<j} \{\vec{x}[j] - \vec{x}[i]\} \right\} \tag{2.89}$$

A variant of the Vandermonde matrix is given. The zeroth column and the last row of the Vandermonde matrix are cancelled.

$$V'[0][n-1] = \begin{bmatrix} \vec{x}[0]^1 & \vec{x}[0]^2 & \vec{x}[0]^3 & ... & \vec{x}[0]^{n-1} \\ \vec{x}[1]^1 & \vec{x}[1]^2 & \vec{x}[1]^3 & ... & \vec{x}[1]^{n-1} \\ \vec{x}[2]^1 & \vec{x}[2]^2 & \vec{x}[2]^3 & ... & \vec{x}[2]^{n-1} \\ ... & ... & ... & ... & ... \\ \vec{x}[n-2]^1 & \vec{x}[n-2]^2 & \vec{x}[n-2]^3 & ... & \vec{x}[n-2]^{n-1} \end{bmatrix} \tag{2.90}$$

The determinant of the variant without powers of Zero is defined.

$$\det(V'[0][n-1]) = \prod^{0\leq i<n-1} \{\vec{x}[i]\} * \prod^{1\leq j<n-1} \left\{ \prod^{0\leq i<j} \{\vec{x}[j] - \vec{x}[i]\} \right\} \tag{2.91}$$

A variant of the Vandermonde matrix is given. The first column and the last row of the Vandermonde matrix are cancelled.

$$V'[1][n-1] = \begin{bmatrix} \vec{x}[0]^0 & \vec{x}[0]^2 & \vec{x}[0]^3 & ... & \vec{x}[0]^{n-1} \\ \vec{x}[1]^0 & \vec{x}[1]^2 & \vec{x}[1]^3 & ... & \vec{x}[1]^{n-1} \\ \vec{x}[2]^0 & \vec{x}[2]^2 & \vec{x}[2]^3 & ... & \vec{x}[2]^{n-1} \\ ... & ... & ... & ... & ... \\ \vec{x}[n-1]^0 & \vec{x}[n-1]^2 & \vec{x}[n-1]^3 & ... & \vec{x}[n-1]^{n-1} \end{bmatrix} \tag{2.92}$$

The determinant of the variant without powers of One is defined.

$$\det(V'[1][n-1]) = \sum^{1\leq j<n-1} \left\{ \sum^{0\leq i<j} \{\vec{x}[j] * \vec{x}[i]\} \right\} * \prod^{1\leq j<n-1} \left\{ \prod^{0\leq i<j} \{\vec{x}[j] - \vec{x}[i]\} \right\} \tag{2.93}$$

Other variants with one or more cancelled rows and columns may be defined.

Chapter 3

Polynomials

$\mathbb{P}$ is the set of all polynomials. A polynomial is a function as sum of terms. A term is a product of a constant and a power of position. A polynomial with a unique term for each exponent is canonical.

$$y = f(\vec{x}) = \sum^{\alpha \in \mathbb{T}\text{nss}(n;s)} \{a[\alpha] * \vec{x}^{\alpha}\}; \qquad f \in \mathbb{P} \tag{3.1}$$

A polynomial of a one-dimensional position is given.

$$f(\vec{x}) = \sum^{\alpha \in \mathbb{T}\text{nss}(1;2)} \{a[\alpha] * \vec{x}^{\alpha}\} = a[0] * \vec{x}^{0} + a[1] * \vec{x}^{1} + a[2] * \vec{x}^{2} \tag{3.2}$$

A polynomial of a two-dimensional position is given.

$$\begin{aligned} f(\vec{x}) = & \sum^{\alpha \in \mathbb{T}\text{nss}(2;2)} \{a[\alpha] * \vec{x}^{\alpha}\} \\ & = a[\langle 0;0\rangle] * \vec{x}^{\langle 0;0\rangle} + a[\langle 0;1\rangle] * \vec{x}^{\langle 0;1\rangle} + a[\langle 1;0\rangle] * \vec{x}^{\langle 1;0\rangle} \\ & \quad + a[\langle 0;2\rangle] * \vec{x}^{\langle 0;2\rangle} + a[\langle 1;1\rangle] * \vec{x}^{\langle 1;1\rangle} + a[\langle 2;0\rangle] * \vec{x}^{\langle 2;0\rangle} \end{aligned} \tag{3.3a}$$

The polynomial is evaluated.

$$\begin{aligned} f\left(\overrightarrow{\langle 3;-4\rangle}\right) = & a[\langle 0;0\rangle] * 3^{0} * (-4)^{0} + a[\langle 0;1\rangle] * 3^{0} * (-4)^{1} + a[\langle 1;0\rangle] * 3^{1} * (-4)^{0} \\ & + a[\langle 0;2\rangle] * 3^{0} * (-4)^{2} + a[\langle 1;1\rangle] * 3^{1} * (-4)^{1} + a[\langle 2;0\rangle] * 3^{2} * (-4)^{0} \end{aligned} \tag{3.3b}$$

3.1 Determining Coefficients by Base Points

An array of unique points is given.

$$\sum^{0 \le i < m} \left\langle \left\langle \vec{X}[i]\,; Y[i] \right\rangle \right\rangle; \qquad m = \text{size}\left(\vec{X}\right) = \text{size}\,(Y) \tag{3.4}$$

A polynomial with an array of m unknown tuples of n dimensions is given. The coefficients are determined by Cramer's Rule [2, p91]. The number of determined coefficients equals the number of points.

$$f(\vec{x}) = \sum^{\alpha\in A} \{a[\alpha] * \vec{x}^{\alpha}\} ; \qquad A \in \mathbb{T}\mathrm{n}(n) ;\ \mathrm{size}\,(A) = m \tag{3.5}$$

The largest possible degree of a complete array with less than m tuples (2.69c) is determined. n is the number of dimensions. m is the number of points or coefficients.

$$p = \mathrm{gSnt}\,(n; m) \qquad q = \binom{n+p}{p} \tag{3.6}$$

The complete array of tuples of a check sum less or equal p is determined.

$$B = \mathbb{T}\mathrm{nss}\,(n; p) \tag{3.7}$$

The complete array of tuples of check sum $(p + 1)$ is given.

$$C = \mathbb{T}\mathrm{ns}\,(n; p + 1) \tag{3.8}$$

All possible combinations of $(m - q)$ tuples (2.73c) are determined. At least one combination exists.

$$D = \mathbb{C}\mathbb{T}\mathrm{n}\,(C; m - q) ; \qquad \mathrm{size}\,(D) > 0 \tag{3.9}$$

If only one combination exists then it equals the complete array of tuples of check sum $(p + 1)$.

$$\mathrm{if}\,(\mathrm{size}\,(D) = 1)\,\mathrm{then}\,(D[0] = C) \tag{3.10}$$

The arrays of tuples that determine possible polynomials are determined.

$$\sum^{0\le j<\mathrm{size}(D)} \langle E[j] = \langle B; D[j]\rangle\rangle \tag{3.11}$$

All possible polynomials are given in terms of unknown coefficients b.

$$\sum^{0\le j<\mathrm{size}(E)} \left\langle g[j](\vec{x}) = \sum^{\alpha\in E[j]} \{b[j][\alpha] * \vec{x}^{\alpha}\} \right\rangle \tag{3.12}$$

The systems of equations are given that may determine the coefficients.

$$\sum^{0\le j<\mathrm{size}(E)} \left\langle \sum^{0\le i<m} \left\langle Y[i] = \sum^{\alpha\in E[j]} \left\{ b[j][\alpha] * \vec{X}[i]^{\alpha} \right\} \right\rangle \right\rangle \tag{3.13}$$

The systems are given in terms of matrices.

$$\sum^{0\le j<\mathrm{size}(E)} \left\langle \sum^{0\le i<m} \left\langle \sum^{\substack{0\le i<m \\ \alpha=E[j][i]}} \left\langle \vec{X}[i]^{\alpha} \right\rangle \right\rangle * \sum^{\substack{0\le i<m \\ \alpha=E[j][i]}} \langle b[j][\alpha]\rangle = \sum^{0\le i<m} \langle Y[i]\rangle \right\rangle \tag{3.14}$$

The base matrix of a possible polynomial is defined.

$$G[j] = \sum^{0\le i<m} \left\langle \sum^{\substack{0\le i<m\\ \alpha=E[j][i]}} \left\langle \vec{X}[i]^{\alpha} \right\rangle \right\rangle \tag{3.15}$$

A polynomial exists if its determinant is not equal Zero.

$$\text{if}\,(\det(G[j]) \neq 0)\,\text{then}\,(g[j] \in \mathbb{P}) \tag{3.16}$$

All determinants equal Zero and thus no polynomial exists if any two points are coincident.

$$\text{if}\left(\vec{X}[i] = \vec{X}[j]\,;\ i \neq j\right)\text{then}\left(\sum^{0\le k<\text{size}(E)} \langle g[k] \notin \mathbb{P} \rangle\right) \tag{3.17}$$

A source matrix follows from a base matrix by substituting one column by the source vector.

$$Q[j][i] = \sum^{0\le i<m} \left\langle \sum^{\substack{0\le i<m\\ \alpha=E[j][i]}} \left\langle \begin{cases} Y[i] & \text{if } i = j \\ \vec{X}[i]^{\alpha} & \text{otherwise} \end{cases} \right\rangle \right\rangle \tag{3.18}$$

The coefficients of a polynomial are determined by Cramer's rule.

$$\sum^{\substack{0\le i<m\\ \alpha=E[j][i]}} \left\langle \text{if}\,(\det(G[j]) \neq 0)\,\text{then}\left(b[j][\alpha] = \frac{\det(Q[j][i])}{\det(G[j])}\right) \right\rangle \tag{3.19}$$

The determinant of the maximal absolute value is given.

$$J = \text{iAbsMax}H; \qquad \sum^{0\le i<m} \langle H[j] = \det(G[j]) \rangle \tag{3.20}$$

A polynomial is given if the determinant of the maximal absolute value is non-zero.

$$\text{if}\,(\det(G[J]) \neq 0)\,\text{then}\left(f(\vec{x}) = g[J](\vec{x})\,; \qquad \sum^{\alpha\in E[J]} \langle a[\alpha] = b[J][\alpha] \rangle\right) \tag{3.21}$$

3.2 Example of One Dimension

A polynomial of an one-dimensional position is determined by three unique points.

$$\sum^{0\le i<3} \langle y[i] = f(\vec{x}[i]) \rangle \tag{3.22}$$

The canonical form is determined.

$$f(\vec{x}) = \sum^{\alpha\in A} \{a[\alpha] * \vec{x}^{\alpha}\}\,; \qquad A \in \mathbb{T}\text{n}(1) \tag{3.23}$$

The smallest degree of an array of tuples of less than three elements equals One and contains two elements.

$$p = \text{gSnt}\,(1;3) = 1; \qquad q = \binom{1+1}{1} = 2 \tag{3.24}$$

The complete array of one-dimensional tuples of a horizontal check sum of less than or equal to One is determined.

$$B = \langle\langle 0\rangle\,;\langle 1\rangle\rangle \tag{3.25}$$

The complete array of tuples of a horizontal check sum of Two is determined.

$$C = \mathbb{T}\text{ns}\,(1;2) = \langle\langle 2\rangle\rangle \tag{3.26}$$

The combination of all arrays with vertical check sums over C is determined.

$$D = \mathbb{CT}\text{n}\,(C;3-2) = \langle\langle\langle 2\rangle\rangle\rangle \tag{3.27}$$

The only array of tuples to determine a polynomial is given.

$$E[0] = \langle B; D[0]\rangle = \langle\langle 0\rangle\,;\langle 1\rangle\,;\langle 2\rangle\rangle \tag{3.28}$$

The terms of the only polynomial are determined.

$$g[0](\vec{x}) = b[0][0] * \vec{x}^0 + b[0][1] * \vec{x}^1 + b[0][2] * \vec{x}^2 \tag{3.29}$$

The coefficients are determined by a system of linear equations.

$$\begin{bmatrix} \vec{x}[0]^0 & \vec{x}[0]^1 & \vec{x}[0]^2 \\ \vec{x}[1]^0 & \vec{x}[1]^1 & \vec{x}[2]^2 \\ \vec{x}[2]^0 & \vec{x}[2]^1 & \vec{x}[1]^2 \end{bmatrix} * \begin{bmatrix} b[0][0] \\ b[0][1] \\ b[0][2] \end{bmatrix} = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \end{bmatrix} \tag{3.30}$$

The polynomial exists since the three points are unique.

$$f(\vec{x}) = g[0](\vec{x})\,; \qquad A = E[0] \tag{3.31}$$

3.3 Example of Two Dimensions

Five unique points of two dimensions are arranged on a the axes of a coordinate system.

i	0	1	2	3	4
$\vec{x}[i][0]$	0	d	0	$-d$	0
$\vec{x}[i][1]$	0	0	d	0	$-d$

(3.32)

The smallest degree of a complete array of tuples with less than five elements equals One.

$$p = \text{gSnt}\,(2;5) = 1; \qquad q = \binom{2+1}{1} = 3; \qquad B = \mathbb{T}\text{nss}\,(2;1) = \langle\langle 0;0\rangle\,;\langle 0;1\rangle\,;\langle 1;0\rangle\rangle \tag{3.33}$$

The complete array of tuples with a horizontal check sum of Two is determined.

$$C = \mathbb{T}\text{ns}\,(n; p+1) = \mathbb{T}\text{ns}\,(2;2) = \langle\langle 0;2\rangle\,;\langle 1;1\rangle\,;\langle 2;0\rangle\rangle \tag{3.34}$$

All combinations of two tuples of a horizontal check sum of Two are determined.

$$D = \mathbb{CT}\text{n}\,(C; m-q) = \big(\ \langle\langle 0;2\rangle\,;\langle 1;1\rangle\rangle\,;\quad \langle\langle 0;2\rangle\,;\langle 2;0\rangle\rangle\,;\quad \langle\langle 1;1\rangle\,;\langle 2;0\rangle\rangle\ \big) \tag{3.35}$$

The resulting arrays of tuples are determined.

$$E[0] = \langle C; D[0]\rangle\,; \qquad E[1] = \langle C; D[1]\rangle\,; \qquad E[2] = \langle C; D[2]\rangle \tag{3.36}$$

A base matrix is determined.

$$G[j] = \begin{bmatrix} 1 & \vec{x}[0]^{\langle 0;1\rangle} & \vec{x}[0]^{\langle 1;0\rangle} & \vec{x}[0]^{E[j][0]} & \vec{x}[0]^{E[j][1]} \\ 1 & \vec{x}[1]^{\langle 0;1\rangle} & \vec{x}[1]^{\langle 1;0\rangle} & \vec{x}[1]^{E[j][0]} & \vec{x}[1]^{E[j][1]} \\ 1 & \vec{x}[2]^{\langle 0;1\rangle} & \vec{x}[2]^{\langle 1;0\rangle} & \vec{x}[2]^{E[j][0]} & \vec{x}[2]^{E[j][1]} \\ 1 & \vec{x}[3]^{\langle 0;1\rangle} & \vec{x}[3]^{\langle 1;0\rangle} & \vec{x}[3]^{E[j][0]} & \vec{x}[3]^{E[j][1]} \\ 1 & \vec{x}[4]^{\langle 0;1\rangle} & \vec{x}[4]^{\langle 1;0\rangle} & \vec{x}[4]^{E[j][0]} & \vec{x}[4]^{E[j][1]} \end{bmatrix} \tag{3.37}$$

Powers of $\langle 1;1\rangle$ equal Zero. Therefore only one base matrix has a determinant of non-zero.

$$\det(G[0]) = 0; \qquad \det(G[1]) \neq 0; \qquad \det(G[2]) = 0 \tag{3.38}$$

The solution is determined by the only available polynomial.

$$J = \text{iAbsMax}(\det(G[0])\,;\det(G[1])\,;\det(G[2])) = 1; \qquad f(\vec{x}) = g[J](\vec{x})\,; \qquad A = E[J] \tag{3.39}$$

3.4 Integer Operations on Polynomials

An operation on polynomials of the same dimensions that results another polynomial on these dimensions is integer. Sums, products and nests of polynomials are discussed.

3.4.1 Sum of Polynomials

An array of tuples is defined.

$$F = \mathbb{T}\text{nss}\,(n; s) \tag{3.40a}$$

A sum of polynomials is given that share the array of tuples. Coefficients may equal Zero.

$$g(\vec{x}) = \sum^{0\le j<N} \{f[j](\vec{x})\} = \sum^{0\le j<N} \left\{ \sum^{\alpha\in F} \{a[j][\alpha] * \vec{x}^{\alpha}\} \right\} \tag{3.40b}$$

The canonical form follows from the transposition of the sums.

$$g(\vec{x}) = \sum^{0\le j<N} \{f[j](\vec{x})\} = \sum^{\alpha\in F} \left\{ \vec{x}^{\alpha} * \sum^{0\le j<N} \{a[j][\alpha]\} \right\} \tag{3.40c}$$

3.4.2 Product of Polynomials

An array of tuples is defined.

$$A = \mathbb{T}\text{nss}\,(n; s) \tag{3.41a}$$

A sum of polynomials is given that share the array of tuples. Coefficients may equal Zero.

$$g(\vec{x}) = \prod^{0\le j<N} \{f[j](\vec{x})\} = \prod^{0\le j<N} \left\{ \sum^{\alpha\in A} \{a[j][\alpha] * \vec{x}^{\alpha}\} \right\} \tag{3.41b}$$

The transposition is given in terms of the N-fold span of the array of tuples (2.77). The exponent of the power of distance is a vertical check sum (2.65). Matrix B may contain more than one array of one vertical check sum since for example $\mathcal{S}\mathbb{T}\,(\langle\alpha;\beta\rangle) = \mathcal{S}\mathbb{T}\,(\langle\beta;\alpha\rangle)$. An array β holds one tuple for each polynomial.

$$g(\vec{x}) = \prod^{0\le j<N} \{f[j](\vec{x})\} = \sum^{\beta\in B} \left\{ \vec{x}^{\mathcal{S}\mathbb{T}(\beta)} * \prod^{0\le j<N} \{a[j][\beta[j]]\} \right\}; \quad B = \text{spancc}\mathbb{T}\text{e}\,(A; N) \tag{3.41c}$$

The matrix of tuples is divided by the vertical check sums (2.80d) in order to obtain the polynomial in canonical form. Array C contains a number of arrays of arrays d of the same vertical check sum. $d[0]$ is always defined since each array of tuples holds at least one element.

$$\prod^{0\le j<N} \{f[j](\vec{x})\} = \sum^{d\in C} \left\{ \vec{x}^{\mathcal{S}\mathbb{T}(d[0])} * \sum^{\beta\in d} \left\{ \prod^{0\le j<N} \{a[j][\beta[j]]\} \right\} \right\}; \quad C = \text{long}\mathbb{T}\mathbb{T}(B) \tag{3.41d}$$

The product of two polynomials is given.

$$\left(a[0][\alpha] * \vec{x}^{\alpha} + a[0][\beta] * \vec{x}^{\beta}\right) * \left(a[1][\alpha] * \vec{x}^{\alpha} + a[1][\beta] * \vec{x}^{\beta}\right) \tag{3.42a}$$
$$= a[0][\alpha] * a[1][\alpha] * \vec{x}^{\alpha+\alpha} + a[0][\beta] * a[1][\alpha] * \vec{x}^{\beta+\alpha} \tag{3.42b}$$
$$+ a[0][\alpha] * a[1][\beta] * \vec{x}^{\alpha+\beta} + a[0][\beta] * a[1][\beta] * \vec{x}^{\beta+\beta} \tag{3.42c}$$
$$= a[0][\alpha] * a[1][\alpha] * \vec{x}^{\alpha+\alpha} + (a[0][\beta] * a[1][\alpha] + a[0][\alpha] * a[1][\beta]) * \vec{x}^{\alpha+\beta} \tag{3.42d}$$
$$+ a[0][\beta] * a[1][\beta] * \vec{x}^{\beta+\beta} \tag{3.42e}$$

3.4.3 Nested Polynomials

Polynomials may be nested arbitrarily. Any element of a position may be a polynomial.

$$\vec{x}[0](\vec{u}) = \sum^{\alpha\in A} \{a[\alpha] * \vec{u}^{\alpha}\}; \qquad A = \mathbb{T}\text{nss}\,(m; M) \tag{3.43a}$$
$$\vec{x}[1](\vec{v}) = \sum^{\beta\in B} \left\{b[\beta] * \vec{v}^{\beta}\right\}; \qquad B = \mathbb{T}\text{nss}\,(n; N) \tag{3.43b}$$

A method of the position of polynomials is given.

$$f(\vec{u}; \vec{v}) = \sum^{\gamma\in C} \{c[\gamma] * \vec{x}^{\gamma}\}; \qquad C = \mathbb{T}\text{nss}\,(2; P) \tag{3.43c}$$

$\vec{v}$ is defined as a polynomial of $\vec{u}$.

$$\vec{v}(u) = \sum^{\delta \in D} \left\{ d[\delta] * \vec{u}^{\delta} \right\}; \qquad D = \mathbb{T}\mathrm{nss}\,(m; Q) \tag{3.43d}$$

Thus method f is a polynomial.

$$f(\vec{u}; \vec{v}(\vec{u})) = g(\vec{u}) \tag{3.43e}$$

Chapter 4

Differentiation Method ∂

A polynomial is given.

$$f(\vec{x}) = \sum^{\alpha\in A} \{a[\alpha] * \vec{x}^{\alpha}\}\,; \qquad A = \mathbb{T}\text{nss}\,(n; m) \tag{4.1}$$

The position is separated. It is geometrically considered a destination as sum of another position and a distance.

$$\vec{x} = \vec{u} + \vec{v} \tag{4.2}$$

The separation turns the polynomial into a method. The value of the method is computed by two variables for each dimension.

$$f(\vec{u} + \vec{v}) = \sum^{\alpha\in A} \{a[\alpha] * (\vec{u} + \vec{v})^{\alpha}\} = g(\vec{u}; \vec{v}) \tag{4.3}$$

The binomial expansion (2.85) is applied by $\vec{v}$. Thus $\vec{u}$ is considered a position and $\vec{v}$ a distance.

$$g(\vec{u}; \vec{v}) = \sum^{\alpha\in A} \left\{ a[\alpha] * \sum_{\beta\leq\alpha}^{\beta\in A} \left\{ \binom{\alpha}{\beta}_{*} * \vec{u}^{\alpha-\beta} * \vec{v}^{\beta} \right\} \right\} \tag{4.4}$$

Another array of tuples is formed in terms of the degree of array A.

$$B = \mathbb{T}\text{nss}\,(n; \mathcal{G}\,(A)) \tag{4.5}$$

The sums are transposed. All factors of only β are put in front of the inner sum to give the Taylor coefficient.

$$g(\vec{u}; \vec{v}) = \sum^{\beta\in B} \left\{ \frac{\vec{v}^{\beta}}{\beta!_{*}} * \sum_{\alpha\geq\beta}^{\alpha\in A} \left\{ a[\alpha] * (\alpha¡\beta)_{*} * \vec{u}^{\alpha-\beta} \right\} \right\} \tag{4.6}$$

The inner sum is defined as the value of the derivative method ∂ of order β and polynomial f. This value or derivative is itself a polynomial of $\vec{u}$. All derivatives of orders greater β equal Zero according to the condition on the sum.

$$\partial\left(\beta; f(\vec{x}); \vec{u}\right) = \sum_{\alpha\geq\beta}^{\alpha\in A} \left\{a[\alpha] * (\alpha¡\beta)_* * \vec{u}^{\alpha-\beta}\right\} = g(\vec{u}) \tag{4.7}$$

An array of all non-zero derivatives is defined.

$$\partial f = \sum^{\beta\in B} \left\langle \partial\left(\beta; f(\vec{x}); \vec{u}\right)\right\rangle \tag{4.8}$$

The body of the derivative of order Zero equals that of the original polynomial.

$$\partial f[\mathbf{0}] = f \tag{4.9}$$

Equation 4.6 is given in terms of Taylor coefficients and derivatives as finite Taylor series of the polynomial.

$$g(\vec{u}; \vec{v}) = \sum^{\beta\in B} \left\{\frac{\vec{v}^{\beta}}{\beta!_*} * \partial f[\beta](\vec{u})\right\} \tag{4.10}$$

A derivative of the origin is given.

$$\partial f[\beta](\mathbf{0}) = \sum_{\alpha\geq\beta}^{\alpha\in A} \left\{a[\alpha] * (\alpha¡\beta)_* * \mathbf{0}^{\alpha-\beta}\right\} = a[\beta] * (\beta¡\beta)_* \tag{4.11}$$

A polynomial coefficient is defined in terms of a derivative of the origin. The descending faculty of (4.11) equals an ascending faculty.

$$a[\beta] = \frac{\partial f[\beta](\mathbf{0})}{\beta!_*} \tag{4.12}$$

A polynomial of two terms is given.

$$f(\vec{x}) = a[\langle 1;0\rangle] * \vec{x}^{\langle 1;0\rangle} + a[\langle 1;2\rangle] * \vec{x}^{\langle 1;2\rangle} \tag{4.13}$$

$$f(\vec{x} = \vec{u}+\vec{v}) = a[\langle 1;0\rangle] * (\vec{u}+\vec{v})^{\langle 1;0\rangle} + a[\langle 1;2\rangle] * (\vec{u}+\vec{v})^{\langle 1;2\rangle} \tag{4.14}$$

$$\begin{aligned} &= \vec{v}^{\langle 0;0\rangle} * \Big(a[\langle 1;0\rangle] * \vec{u}^{\langle 1;0\rangle} + a[\langle 1;2\rangle] * \vec{u}^{\langle 1;2\rangle}\Big) + \vec{v}^{\langle 0;1\rangle} * \Big(2 * a[\langle 1;2\rangle] * \vec{u}^{\langle 1;1\rangle}\Big) \\ &+ \vec{v}^{\langle 1;0\rangle} * \Big(a[\langle 1;0\rangle] + a[\langle 1;2\rangle] * \vec{u}^{\langle 0;1\rangle}\Big) + \frac{1}{2} * \vec{v}^{\langle 0;2\rangle} * \Big(2 * a[\langle 1;2\rangle] * \vec{u}^{\langle 1;0\rangle}\Big) \\ &+ \vec{v}^{\langle 1;1\rangle} * \Big(2 * a[\langle 1;2\rangle] * \vec{u}^{\langle 0;1\rangle}\Big) + \frac{1}{2} * \vec{v}^{\langle 1;2\rangle} * \Big(2 * a[\langle 1;2\rangle] * \vec{u}^{\langle 0;0\rangle}\Big) \end{aligned} \tag{4.15}$$

4.1 Derivatives of Derivatives

Position $\vec{u}$ of (4.6) is separated.

$$\vec{u} = \vec{s} + \vec{t} \tag{4.16}$$

The separation is substituted into (4.6).

$$A = \mathbb{T}\text{nss}\,(n;m)\,; \qquad B = \mathbb{T}\text{nss}\,(d;\mathcal{G}\,(A)) \tag{4.17}$$

$$f\left(\vec{u} = \vec{s} + \vec{t}; \vec{v}\right) = \sum^{\beta\in B} \left\{ \frac{\vec{v}^{\beta}}{\beta!_*} * \sum_{\alpha\geq\beta}^{\alpha\in A} \left\{ a[\alpha] * (\alpha¡\beta)_* * \left(\vec{s}+\vec{t}\right)^{\alpha-\beta} \right\} \right\} = f\left(\vec{s};\vec{t};\vec{v}\right) \tag{4.18}$$

The binomial expansion is applied by $\vec{t}$.

$$f\left(\vec{s};\vec{t};\vec{v}\right) = \sum^{\beta\in B} \left\{ \frac{\vec{v}^{\beta}}{\beta!_*} * \sum_{\alpha\geq\beta}^{\alpha\in A} \left\{ a[\alpha] * (\alpha¡\beta)_* * \sum_{\gamma\leq\alpha-\beta}^{\gamma\in A} \left\{ \binom{\alpha-\beta}{\gamma} * \vec{s}^{\alpha-\beta-\gamma} * \vec{t}^{\gamma} \right\} \right\} \right\} \tag{4.19}$$

The inner two sums are transposed and another Taylor coefficient is formed.

$$f\left(\vec{s};\vec{t};\vec{v}\right) = \sum^{\beta\in B} \left\{ \frac{\vec{v}^{\beta}}{\beta!_*} * \sum^{\gamma\in B} \left\{ \frac{\vec{t}^{\gamma}}{\gamma!_*} * \sum_{\alpha\geq\beta+\gamma}^{\alpha\in A} \left\{ a[\alpha] * (\alpha¡\beta)_* * ((\alpha-\beta)\,¡\gamma)_* * \vec{s}^{\alpha-\beta-\gamma} \right\} \right\} \right\} \tag{4.20}$$

The descending faculties simplify.

$$(\alpha¡\beta)_* * ((\alpha-\beta)\,¡\gamma)_* = (\alpha¡\,(\beta+\gamma))_* \tag{4.21}$$

The expression is substituted into (4.20) and the exponent of that inner sum is rewritten.

$$f\left(\vec{s};\vec{t};\vec{v}\right) = \sum^{\beta\in B} \left\{ \frac{\vec{v}^{\beta}}{\beta!_*} * \sum^{\gamma\in B} \left\{ \frac{\vec{t}^{\gamma}}{\gamma!_*} * \sum_{\alpha\geq\beta+\gamma}^{\alpha\in A} \left\{ a[\alpha] * (\alpha¡\,(\beta+\gamma))_* * \vec{s}^{\alpha-(\beta+\gamma)} \right\} \right\} \right\} \tag{4.22}$$

The derivative of order γ of a derivative of order β of a polynomial equals the derivative of order $\beta+\gamma$ of that polynomial.

$$\partial\,(\gamma;\partial\,(\beta;f(\vec{x})\,;\vec{u})\,;\vec{s}) = \partial\,(\beta+\gamma;f(\vec{x})\,;\vec{s}) \tag{4.23}$$

4.2 Integer Rules for Differentiation

4.2.1 Differentiating a Sum

A sum of polynomials is given that share an array of tuples. Coefficients may equal Zero.

$$g(\vec{x}) = \sum^{0\leq j<N} \{f[j](\vec{x})\} = \sum^{0\leq j<N} \left\{ \sum^{\alpha\in A} \{a[j][\alpha] * \vec{x}^{\alpha}\} \right\}; \qquad A = \mathbb{T}\text{nss}\,(n;s) \tag{4.24}$$

Another array of tuples is formed in terms of the degree of array A.

$$B = \mathbb{T}\text{nss}\,(n;\mathcal{G}\,(A)) \tag{4.25}$$

The derivatives of the polynomial are determined.

$$\sum^{\beta\in B}\left\langle\sum^{0\leq j<N}\left\langle\partial f[j][\beta](\vec{u})=\partial\left(\beta;f[j](\vec{x})\,;\vec{u}\right)\right\rangle\right\rangle \tag{4.26}$$

The derivative method is applied.

$$h(\vec{u};\vec{v})=\sum^{0\leq j<N}\left\{\sum^{\beta\in B}\left\{\frac{\vec{v}^{\beta}}{\beta!_*}*\partial f[j][\beta](\vec{u})\right\}\right\} \tag{4.27}$$

The sums are transposed.

$$h(\vec{u};\vec{v})=\sum^{\beta\in B}\left\{\frac{\vec{v}^{\beta}}{\beta!_*}*\sum^{0\leq j<N}\left\{\partial f[j][\beta](\vec{u})\right\}\right\} \tag{4.28}$$

The Taylor coefficient does not depend on the number of polynomials. Thus the derivative of order β of a sum of polynomials is defined as the sum of derivatives of the same order of the individual polynomials.

$$\partial g[\beta](\vec{u})=\sum^{0\leq j<N}\left\{\partial f[j][\beta](\vec{u})\right\} \tag{4.29}$$

4.2.2 Differentiating a Product

A product of polynomials is given that share an array of tuples. Coefficients may equal Zero.

$$g(\vec{x})=\prod^{0\leq j<N}\left\{f[j](\vec{x})\right\}=\prod^{0\leq j<N}\left\{\sum^{\alpha\in A}\left\{a[j][\alpha]*\vec{x}^{\alpha}\right\}\right\};\qquad A=\mathbb{T}\mathrm{nss}\left(n;s\right) \tag{4.30}$$

Another array of tuples is formed in terms of the degree of array A.

$$B=\mathbb{T}\mathrm{nss}\left(n;\mathcal{G}\left(A\right)\right) \tag{4.31}$$

The derivatives of the polynomial are determined.

$$\sum^{\beta\in B}\left\langle\sum^{0\leq j<N}\left\langle\partial f[j][\beta](\vec{u})=\partial\left(\beta;f[j](\vec{x})\,;\vec{u}\right)\right\rangle\right\rangle \tag{4.32}$$

The derivative method is applied.

$$h(\vec{u};\vec{v})=\prod^{0\leq j<N}\left\{\sum^{\beta\in B}\left\{\frac{\vec{v}^{\beta}}{\beta!_*}*\partial f[j][\beta](\vec{u})\right\}\right\} \tag{4.33}$$

The N-fold span of the array of tuples B is determined (2.77). The exponent of the power of distance is a vertical check sum (2.65). Matrix C may contain more than one array of one vertical check sum since for example $\mathcal{S}\mathbb{T}(\langle\alpha;\beta\rangle) = \mathcal{S}\mathbb{T}(\langle\beta;\alpha\rangle)$. An array d holds one tuple for each polynomial.

$$\prod^{0\leq j<N}\{f[j](\vec{x})\} = \sum^{c\in C}\left\{\vec{v}^{\mathcal{S}\mathbb{T}(\gamma)} * \prod^{0\leq j<N}\left\{\frac{\partial f[j][c[j]](\vec{u})}{c[j]!_*}\right\}\right\}; \qquad C = \mathrm{spancc}\mathbb{T}\mathrm{e}(B;N) \tag{4.34}$$

The matrix of tuples is divided by the vertical check sums (2.80d). Array D contains a number of arrays of arrays d of the same vertical check sum. $d[0]$ is always defined since each array of tuples holds at least one element.

$$\prod^{0\leq j<N}\{f[j](\vec{x})\} = \sum^{d\in D}\left\{\vec{v}^{\mathcal{S}\mathbb{T}(d[0])} * \sum^{c\in d}\left\{\prod^{0\leq j<N}\left\{\frac{\partial f[j][c[j]](\vec{u})}{c[j]!_*}\right\}\right\}\right\}; \quad D = \mathrm{long}\mathbb{T}\mathbb{T}(C) \tag{4.35}$$

A quotient of vertical check sums is introduced in order to obtain the Taylor coefficient according to (4.6).

$$\prod^{0\leq j<N}\{f[j](\vec{x})\} = \sum^{d\in D}\left\{\frac{\vec{v}^{\mathcal{S}\mathbb{T}(d[0])}}{\mathcal{S}\mathbb{T}(d[0])!_*} * \mathcal{S}\mathbb{T}(d[0])!_* * \sum^{c\in d}\left\{\prod^{0\leq j<N}\left\{\frac{\partial f[j][c[j]](\vec{u})}{c[j]!_*}\right\}\right\}\right\} \tag{4.36}$$

The derivative of a product of polynomials is determined with an array of arrays of the same vertical check sum (2.74).

$$\partial\left(\beta;\prod^{0\leq j<N}\{f[j](\vec{x})\};\vec{u}\right) = \beta!_* * \sum^{c\in E}\left\{\prod^{0\leq j<N}\left\{\frac{\partial f[j][c[j]](\vec{u})}{c[j]!_*}\right\}\right\}; \quad E = \mathrm{long}\alpha\mathrm{n}(\beta;N) \tag{4.37}$$

Derivatives of a checksum equal One result the product rule of differentiation for polynomials.

$$\mathcal{S}(\beta) = 1; \qquad E = \mathrm{long}\alpha\mathrm{n}(\beta;N) \tag{4.38a}$$

$$\partial\left(\beta;\prod^{0\leq j<N}\{f[j](\vec{x})\};\vec{u}\right) = \sum^{\gamma\in E}\left\{\sum^{0\leq k<N}\left\{\partial f[k][\beta](\vec{u}) * \prod_{j\neq k}^{0\leq j<N}\{\partial f[j][\mathbf{0}](\vec{u})\}\right\}\right\} \tag{4.38b}$$

Examples

A product of two polynomials is given.

$$g(\vec{x}) = f[0](\vec{x}) * f[1](\vec{x}) \tag{4.39a}$$

The derivative of a factor and the product are determined.

$$\partial(\beta;f[j](\vec{x});\vec{u}) = \partial f[j][\beta](\vec{u}) \tag{4.39b}$$

$$\partial(\beta;g(\vec{x});\vec{u}) = \partial g[\beta](\vec{u}) \tag{4.39c}$$

A product rule is determined for a position of two dimensions.

$$E = \mathrm{long}\alpha\mathrm{n}(\langle 0;1\rangle;2) = (\quad \langle\langle 0;0\rangle;\langle 0;1\rangle\rangle; \qquad \langle\langle 0;1\rangle;\langle 0;0\rangle\rangle \quad) \tag{4.39d}$$

$$\partial g[\langle 0;1\rangle](\vec{u}) = \partial f[0][\langle 0;0\rangle](\vec{u}) * \partial f[1][\langle 0;1\rangle](\vec{u}) + \partial f[0][\langle 0;1\rangle](\vec{u}) * \partial f[1][\langle 0;0\rangle](\vec{u}) \tag{4.39e}$$

The order of multiplication and differentiation is not significant. Two polynomials are given.

$$f(\vec{x}) = a[0] * \vec{x}^0 + a[1] * \vec{x}^1 + a[2] * \vec{x}^2 \tag{4.40a}$$
$$g(\vec{x}) = b[0] * \vec{x}^0 + b[1] * \vec{x}^1 + b[2] * \vec{x}^2 \tag{4.40b}$$

The product is determined.

$$\begin{aligned} h(\vec{x}) = f(\vec{x}) * g(\vec{x}) = {} & a[0] * b[0] * \vec{x}^0 + a[0] * b[1] * \vec{x}^1 + a[0] * b[2] * \vec{x}^2 \\ & + a[1] * b[0] * \vec{x}^1 + a[1] * b[1] * \vec{x}^2 + a[1] * b[2] * \vec{x}^3 \\ & + a[2] * b[0] * \vec{x}^2 + a[2] * b[1] * \vec{x}^3 + a[2] * b[2] * \vec{x}^4 \end{aligned} \tag{4.40c}$$

The only second derivative is determined.

$$\begin{aligned} \partial\left(2; h(\vec{x}); \vec{u}\right) = {} & 2 * a[0] * b[2] * \vec{u}^0 + 2 * a[1] * b[1] * \vec{x}^0 + 6 * a[1] * b[2] * \vec{x}^1 \\ & + 2 * a[2] * b[0] * \vec{x}^0 + 6 * a[2] * b[1] * \vec{x}^1 + 12 * a[2] * b[2] * \vec{x}^2 \end{aligned} \tag{4.40d}$$

A matrix of tuples is determined.

$$E = \text{long}\alpha\text{n}(2; 2) = \left(\quad \langle\langle 0\rangle ; \langle 2\rangle\rangle ; \quad \langle\langle 1\rangle ; \langle 1\rangle\rangle ; \quad \langle\langle 2\rangle ; \langle 0\rangle\rangle \quad \right) \tag{4.40e}$$

The derivatives of the factors are determined.

$$\partial\left(\beta; f(\vec{x}); \vec{u}\right) = f[\beta](\vec{u}) \tag{4.40f}$$
$$\partial\left(\beta; g(\vec{x}); \vec{u}\right) = g[\beta](\vec{u}) \tag{4.40g}$$

The derivative of the only second order of the product is determined.

$$\frac{2!}{0!2!} * f[0](\vec{u}) * g[2](\vec{u}) + \frac{2!}{1!1!} * f[1](\vec{u}) * g[1](\vec{u}) + \frac{2!}{2!0!} * f[2](\vec{u}) * g[0](\vec{u}) \tag{4.40h}$$
$$\begin{aligned} = {} & \left(a[0] * \vec{u}^0 + a[1] * \vec{u}^1 + a[2] * \vec{u}^2\right) * \left(2 * b[2] * \vec{u}^0\right) \\ & + 2 * \left(a[1]\, \vec{u}^0 + 2 * a[2] * \vec{u}^1\right) * \left(b[1] * \vec{u}^0 + 2 * b[2] * \vec{u}^1\right) \\ & + \left(2 * a[2] * \vec{u}^0\right) * \left(b[0] * \vec{u}^0 + b[1] * \vec{u}^1 + b[2] * \vec{u}^2\right) \end{aligned} \tag{4.40i}$$
$$= \partial\left(2; h(\vec{x}); \vec{u}\right) \tag{4.40j}$$

4.2.3 Derivatives of a Polynomial of a Polynomial

The formulation of all derivatives of arbitrarily nested polynomials is unknown. The derivatives of a polynomial of a single polynomial is considered here only to a point where it becomes non-trivial. However, the chain rule is given which allows for determining a first order derivative. This rule may be applied repeatedly to obtain derivatives of higher orders.

General Formulation

A polynomial g of another polynomial f is given.

$$f(\vec{x}) = \sum^{0 \le j < N} \left\{ b[j] * \left(\sum^{\alpha \in A} \left\{ a[\alpha] * \vec{x}^\alpha \right\} \right) \right\}; \qquad A = \mathbb{T}\text{nss}\left(n; s\right) \tag{4.41}$$

$$g(f(\vec{x})) = \sum^{0 \le j < N} \left\{ b[j] * \left(f(\vec{x})\right)^j \right\} = g(\vec{x}) \tag{4.42}$$

The derivative method is applied to the inner polynomial. An array of all derivatives is obtained according to (4.8).

$$h(\vec{u};\vec{v}) = \sum^{0\le j<N} \left\{ b[j] * \left(\sum^{\beta\in B} \left\{ \frac{\vec{v}^{\beta}}{\beta!_*} * \partial f[\beta](\vec{u}) \right\} \right)^{j} \right\}; \qquad B = \mathbb{T}\mathrm{nss}\,(n;\mathcal{G}\,(A)) \tag{4.43}$$

The power is expressed as a product.

$$h(\vec{u};\vec{v}) = \sum^{0\le j<N} \left\{ b[j] * \prod^{0\le i<j} \left\{ \sum^{\beta\in B} \left\{ \frac{\vec{v}^{\beta}}{\beta!_*} * \partial f[\beta](\vec{u}) \right\} \right\} \right\} \tag{4.44}$$

The inner sums are transposed according to an N-fold span (2.77)

$$C[j] = \mathrm{spancc}\mathbb{T}\mathrm{e}\,(B;j) \tag{4.45}$$

$$h(\vec{u};\vec{v}) = \sum^{0\le j<N} \left\{ b[j] * \sum^{\gamma\in C[j]} \left\{ \vec{v}^{\mathcal{S}\mathbb{T}(\gamma)} * \prod^{0\le i<j} \left\{ \frac{\partial f[\gamma[i]](\vec{u})}{\gamma[i]!_*} \right\} \right\} \right\} \tag{4.46}$$

The matrix of tuples is divided by vertical check sums (2.80d).

$$D[j] = \mathrm{long}\mathbb{T}\mathbb{T}(C[j]) \tag{4.47}$$

$$h(\vec{u};\vec{v}) = \sum^{0\le j<N} \left\{ b[j] * \sum^{\delta\in D[j]} \left\{ \vec{v}^{\mathcal{S}\mathbb{T}(\delta[0])} * \sum^{\gamma\in\delta} \left\{ \prod^{0\le i<j} \left\{ \frac{\partial f[\gamma[i]](\vec{u})}{\gamma[i]!_*} \right\} \right\} \right\} \right\} \tag{4.48}$$

The formulation of the transposition of the outer sums requires an array of terms, see example below. These terms are composed of coefficients and powers of distance which makes the formulation non-trivial.

Example

A polynomial of a polynomial is given.

$$\alpha = \langle 0;0\rangle\,; \qquad \beta = \langle 0;1\rangle\,; \qquad \gamma = \langle 1;0\rangle \tag{4.49}$$

$$f(\vec{x}) = a[\alpha] * \vec{x}^{\alpha} + a[\beta] * \vec{x}^{\beta} + a[\gamma] * \vec{x}^{\gamma} \tag{4.50}$$

$$g(f(\vec{x})) = b[0] * f^{\langle 0\rangle} + b[1] * f^{\langle 1\rangle} + b[2] * f^{\langle 2\rangle} \tag{4.51}$$

The separation of the position results three non-zero Taylor terms for the inner polynomial.

$$t[\alpha] = \frac{\vec{v}^{\alpha}}{\alpha!_*} * \partial f[\alpha](\vec{u})\,; \qquad t[\beta] = \frac{\vec{v}^{\beta}}{\beta!_*} * \partial f[\beta](\vec{u})\,; \qquad t[\gamma] = \frac{\vec{v}^{\gamma}}{\gamma!_*} * \partial f[\gamma](\vec{u}) \tag{4.52}$$

The separation turns the outer polynomial into a method.

$$g(\vec{u}+\vec{v}) = h(\vec{u};\vec{v}) = b[0] + b[1] * (t[\alpha] + t[\beta] + t[\gamma]) + b[2] * (t[\alpha] + t[\beta] + t[\gamma])^2 \tag{4.53}$$

The two-fold span is determined.

$$h(\vec{u};\vec{v}) = b[0] + b[1]*(t[\alpha]+t[\beta]+t[\gamma]) + b[2]*\begin{pmatrix} t[\alpha]*t[\alpha]+t[\alpha]*t[\beta]+t[\alpha]*t[\gamma] \\ +\,t[\beta]*t[\alpha]+t[\beta]*t[\beta]+t[\beta]*t[\gamma] \\ +\,t[\gamma]*t[\alpha]+t[\gamma]*t[\beta]+t[\gamma]*t[\gamma] \end{pmatrix} \quad (4.54)$$

The division by vertical check sums is applied.

$$h(\vec{u};\vec{v}) = b[0] + b[1]*\begin{pmatrix} \langle t[\alpha]\rangle \\ +\langle t[\beta]\rangle \\ +\langle t[\gamma]\rangle \end{pmatrix} + b[2]*\begin{pmatrix} \langle t[\alpha]*t[\alpha]\rangle \\ +\langle t[\alpha]*t[\beta]+t[\beta]*t[\alpha]\rangle \\ +\langle t[\alpha]*t[\gamma]+t[\gamma]*t[\alpha]\rangle \\ +\langle t[\beta]*t[\beta]\rangle \\ +\langle t[\beta]*t[\gamma]+t[\gamma]*t[\beta]\rangle \\ +\langle t[\gamma]*t[\gamma]\rangle \end{pmatrix} \quad (4.55)$$

The terms are transposed and result the derivatives of all orders. This step is non-trivial since an analytical formulation requires composed entities.

$$h(\vec{u};\vec{v}) = \begin{pmatrix} (b[0]+b[1]*(t[\alpha])+b[2]*(t[\alpha]*t[\alpha])) \\ +\,(b[1]*(t[\beta])+b[2]*(t[\alpha]*t[\beta]+t[\beta]*t[\alpha])) \\ +\,(b[1]*(t[\gamma])+b[2]*(t[\alpha]*t[\gamma]+t[\gamma]*t[\alpha])) \\ +\,(b[2]*(t[\beta]*t[\beta])) \\ +\,(b[2]*(t[\beta]*t[\gamma]+t[\gamma]*t[\beta])) \\ +\,(b[2]*(t[\gamma]*t[\gamma])) \end{pmatrix} \quad (4.56)$$

4.2.4 Derivatives of First Order

A polynomial of a single polynomial of an n-dimensional position is given.

$$f(\vec{x}) = \sum^{0\le j<N}\left\{b[j]*\left(\sum^{\alpha\in A}\{a[\alpha]*\vec{x}^{\alpha}\}\right)\right\}; \qquad A = \mathbb{T}\mathrm{nss}\,(n;s) \quad (4.57)$$

$$g(f(\vec{x})) = \sum^{0\le j<N}\left\{b[j]*(f(\vec{x}))^{j}\right\} = g(\vec{x}) \quad (4.58)$$

The derivative method is applied to the inner polynomial. An array of all non-zero derivatives is determined according to (4.8).

$$h(\vec{u};\vec{v}) = \sum^{0\le j<N}\left\{b[j]*\left(\sum^{\beta\in B}\left\{\frac{\vec{v}^{\beta}}{\beta!_*}*\partial f[\beta](\vec{u})\right\}\right)^{j}\right\}; \qquad B = \mathbb{T}\mathrm{nss}\,(n;\mathcal{G}\,(A)) \quad (4.59)$$

The linear Taylor terms are separated. The zeroth order Taylor coefficient equals One and is not denoted.

$$h(\vec{u};\vec{v}) = \sum^{0\le j<N}\left\{b[j]*\left(\partial f[\mathbf{0}](\vec{u}) + \sum_{\mathcal{S}(\beta)=1}^{\beta\in B}\left\{\frac{\vec{v}^{\beta}}{\beta!_*}*\partial f[\beta](\vec{u})\right\} + \sum_{\mathcal{S}(\beta)\ge 2}^{\beta\in B}\left\{\frac{\vec{v}^{\beta}}{\beta!_*}*\partial f[\beta](\vec{u})\right\}\right)^{j}\right\} \quad (4.60)$$

All Taylor terms of orders greater than One are of no further interest and denoted by a remainder.

$$h(\vec{u};\vec{v}) = \sum^{0\le j<N}\left\{b[j]*\left(\partial f[\mathbf{0}](\vec{u}) + \sum_{\mathcal{S}(\beta)=1}^{\beta\in B}\left\{\frac{\vec{v}^{\beta}}{\beta!_*}*\partial f[\beta](\vec{u})\right\}\right)^{j}\right\} + r[2][0] \quad (4.61)$$

The binomial expansion is applied by the first order Taylor terms.

$$h(\vec{u};\vec{v}) = \sum^{0\leq j<N} \left\{ b[j]* \sum^{0\leq k<j} \left\{ \binom{j}{k} *(\partial f[\mathbf{0}](\vec{u}))^{j-k} * \left(\sum_{\mathcal{S}(\beta)=1}^{\beta\in B} \left\{ \frac{\vec{v}^{\beta}}{\beta!_*} * \partial f[\beta](\vec{u}) \right\} \right)^{k} \right\} \right\} + r[2][0] \tag{4.62}$$

The outer sums are transposed.

$$h(\vec{u};\vec{v}) = \sum^{0\leq k<N} \left\{ \sum^{k\leq j<N} \left\{ b[j]* \binom{j}{k} * (\partial f[\mathbf{0}](\vec{u}))^{j-k} * \left(\sum_{\mathcal{S}(\beta)=1}^{\beta\in B} \left\{ \frac{\vec{v}^{\beta}}{\beta!_*} * \partial f[\beta](\vec{u}) \right\} \right)^{k} \right\} \right\} + r[2][0] \tag{4.63}$$

The outermost sum is resolved into the first two cases $k = 0$, $k = 1$ and another remainder of expressions of Taylor terms of orders greater than One.

$$\begin{aligned} h(\vec{u};\vec{v}) = & \sum^{0\leq j<N} \left\{ b[j]*(\partial f[\mathbf{0}](\vec{u}))^{j} \right\} + r[2][0] + r[2][1] \\ & + \sum^{1\leq j<N} \left\{ j * b[j]*(\partial f[\mathbf{0}](\vec{u}))^{j-1} * \sum_{\mathcal{S}(\beta)=1}^{\beta\in B} \left\{ \frac{\vec{v}^{\beta}}{\beta!_*} * \partial f[\beta](\vec{u}) \right\} \right\} \end{aligned} \tag{4.64}$$

The double sum is transposed.

$$\begin{aligned} h(\vec{u};\vec{v}) = & \sum^{0\leq j<N} \left\{ b[j]*(\partial f[\mathbf{0}](\vec{u}))^{j} \right\} + r[2][0] + r[2][1] \\ & + \sum_{\mathcal{S}(\beta)=1}^{\beta\in B} \left\{ \frac{\vec{v}^{\beta}}{\beta!_*} * \partial f[\beta](\vec{u}) * \sum^{1\leq j<N} \left\{ j * b[j]*(\partial f[\mathbf{0}](\vec{u}))^{j-1} \right\} \right\} \end{aligned} \tag{4.65}$$

The value of the inner sum is defined as outer derivative of first order.

$$\partial g[1](f(\vec{u})) = \sum^{1\leq j<N} \left\{ j * b[j]*(\partial f[\mathbf{0}](\vec{u}))^{j-1} \right\} \tag{4.66}$$

The definition is substituted into (4.65).

$$\begin{aligned} h(\vec{u};\vec{v}) = & \sum^{0\leq j<N} \left\{ b[j]*(f[\mathbf{0}](\vec{u}))^{j} \right\} + r[2][0] + r[2][1] \\ & + \sum_{\mathcal{S}(\beta)=1}^{\beta\in B} \left\{ \frac{\vec{v}^{\beta}}{\beta!_*} * \partial f[\beta](\vec{u}) * \partial g[1](f(\vec{u})) \right\} \end{aligned} \tag{4.67}$$

A derivative of a horizontal check sum equal One of a polynomial of another polynomial is defined. It is the product of the first order derivatives.

$$\mathcal{S}(\beta) = 1; \qquad \partial g[\beta](f(\vec{u})) = \partial g[1](f(\vec{u})) * \partial f[\beta](\vec{u}) \tag{4.68}$$

Derivatives of higher orders are obtained by repeated differentiation. A chain of first order derivatives follows for repeatedly nested polynomials.

4.3 Specific Conditions of Derivatives

The Dirichlet condition determines a point.

$$y = \partial f[\mathbf{0}]\left(\vec{U}\right) = f\left(\vec{U}\right) \tag{4.69}$$

A von-Neumann condition determines the total differential in terms of a position and a distance.

$$b = \sum^{\beta \in B} \left\{ \vec{V}^{\beta} * \partial f[\beta]\left(\vec{U}\right) \right\}; \qquad B = \mathbb{T}\mathrm{ns}\,(n; 1) \tag{4.70}$$

A Poisson condition determines the sum of all non-mixed derivatives of second order.

$$c = \sum^{\beta \in B} \left\{ \partial f[\beta + \beta]\left(\vec{U}\right) \right\}; \qquad B = \mathbb{T}\mathrm{ns}\,(n; 1) \tag{4.71}$$

4.4 Determining Coefficients by Derivatives

A polynomial is determined by differential expressions of a position $\vec{u}$. These expressions are given in terms of constant points. Therefore a polynomial of any other position such as $\vec{x}$ is determined consistently by these conditions.

A polynomial and its significant derivatives are determined.

$$f(\vec{x}) = \sum^{\alpha \in A} \{a[\alpha] * \vec{x}^{\alpha}\} \qquad A \in \mathbb{T}\mathrm{n}(n) \tag{4.72}$$

$$\sum^{\beta \in B} \langle \partial f[\beta](\vec{u}) = \partial\,(\beta; f(\vec{x})\,; \vec{u}) \rangle \qquad B = \mathbb{T}\mathrm{nss}\,(n; \mathcal{G}\,(A)) \tag{4.73}$$

An array of unique constant positions is determined.

$$\sum^{0 \le j < J} \left\langle \vec{U}[j] = \mathrm{const} \right\rangle \tag{4.74}$$

A differential term is a product of a constant and a derivative. A differential equation is a sum of differential terms. A complete differential equation combines all derivatives at all locations. The constant of a term may equal Zero such that a term is insignificant. An array of complete differential equations on an array of unique locations is determined.

$$\sum^{0 \le k < K} \left\langle \sum^{0 \le j < J} \left\{ \sum^{\beta \in B} \left\{ b[k][j][\beta] * \partial f[\beta]\left(\vec{U}[j]\right) \right\} \right\} = c[k] \right\rangle; \quad b[k][j][\beta] \in \mathbb{R};\ c[k] \in \mathbb{R} \tag{4.75}$$

The K differential equations determine K coefficients of a polynomial if such a polynomial exists. A polynomial requires an array of tuples. The largest possible degree of a complete array with less than K tuples (2.69c) is determined.

$$p = \mathrm{gSnt}\,(n; K) \tag{4.76}$$

The complete array of tuples of a horizontal check sum less than or equal to p is determined.

$$C = \mathbb{T}\text{nss}\,(n; p) \tag{4.77}$$

The complete array of tuples of a horizontal check sum of $(p+1)$ is determined. This array holds at least one element.

$$D = \mathbb{CT}\text{n}\left(\mathbb{T}\text{ns}\,(n; p+1)\,; K - \binom{n+p}{p}\right); \qquad L = \text{size}\,(D)\,;\ L > 0 \tag{4.78}$$

L arrays of K tuples (2.73c) are determined that may determine L polynomials.

$$\sum^{0 \le l < L} \left\langle\ E[l] = \langle C; D[l] \rangle\ \right\rangle \tag{4.79}$$

The degrees of all arrays are equal and the complete array of tuples of this degree equals B of (4.73).

$$\sum^{0 \le l < L} \left\langle\ \mathbb{T}\text{nss}\,(n; \mathcal{G}\,(E[l])) = B\ \right\rangle \tag{4.80}$$

The possible polynomials are given with unknown coefficients d.

$$\sum^{0 \le l < L} \left\langle g[l](\vec{x}) = \sum^{\alpha \in E[l]} \{d[l][\alpha] * \vec{x}^{\alpha}\} \right\rangle \tag{4.81}$$

A matrix of differential equations is determined with K equations for each of L polynomials.

$$\sum^{0 \le l < L} \left\langle \sum^{0 \le k < K} \left\langle \sum^{0 \le j < J} \left\{ \sum^{\beta \in B} \left\{ b[k][j][\beta] * \partial g[l][\beta]\left(\vec{U}[j]\right) \right\} \right\} = c[k] \right\rangle \right\rangle \tag{4.82}$$

The derivatives are denoted explicitly.

$$\sum^{0 \le l < L} \left\langle \sum^{0 \le k < K} \left\langle \sum^{0 \le j < J} \left\{ \sum^{\beta \in B} \left\{ b[k][j][\beta] * \sum_{\alpha \ge \beta}^{\alpha \in E[l]} \left\{ d[l][\alpha] * (\alpha¡\beta)_* * \vec{U}[j]^{\alpha - \beta} \right\} \right\} \right\} = c[k] \right\rangle \right\rangle \tag{4.83}$$

The equations are rearranged such that all constants are elements of the innermost sum.

$$\sum^{0 \le l < L} \left\langle \sum^{0 \le k < K} \left\langle \sum^{\alpha \in E[l]} \left\{ d[l][\alpha] * \sum^{0 \le j < J} \left\{ \sum_{\beta \le \alpha}^{\beta \in B} \left\{ b[k][j][\beta] * (\alpha¡\beta)_* * \vec{U}[j]^{\alpha - \beta} \right\} \right\} \right\} = c[k] \right\rangle \right\rangle \tag{4.84}$$

The constants are equal for all polynomials and determined by a method of two parameters.

$$e(k; \alpha) = \sum^{0 \le j < J} \left\{ \sum_{\beta \le \alpha}^{\beta \in B} \left\{ b[k][j][\beta] * (\alpha¡\beta)_* * \vec{U}[j]^{\alpha - \beta} \right\} \right\} \tag{4.85}$$

The method is substituted into the differential equations.

$$\sum^{0\leq l<L} \left\langle \sum^{0\leq k<K} \left\langle \sum^{\alpha\in E[l]} \left\{ \; d[l][\alpha] * e(k;\alpha) \; \right\} = c[k] \right\rangle \right\rangle \tag{4.86}$$

The arrays of tuples have as many elements as the array of differential equations.

$$\sum^{0\leq l<L} \langle \mathrm{size}\,(E[l]) = K \rangle \tag{4.87}$$

The equations of each polynomial are denoted in terms of matrices.

$$\sum^{0\leq l<L} \left\langle \sum^{0\leq k<K} \left\langle \sum^{\substack{0\leq i<K \\ \alpha=E[l][i]}} \left\langle \; e(k;\alpha) \; \right\rangle \right\rangle * \sum^{\substack{0\leq i<K \\ \alpha=E[l][i]}} \left\langle \; d[l][\alpha] \; \right\rangle = \sum^{0\leq k<K} \left\langle \; c[k] \; \right\rangle \right\rangle \tag{4.88}$$

An array of base matrices is determined.

$$\sum^{0\leq l<L} \left\langle G[l] = \sum^{0\leq k<K} \left\langle \sum^{\substack{0\leq i<K \\ \alpha=F[l][i]}} \left\langle \; e(k;\alpha) \; \right\rangle \right\rangle \right\rangle \tag{4.89}$$

A source matrix results from a base matrix by substituting one column by the source c which is equal for all polynomials.

$$\sum^{0\leq l<L} \left\langle \sum^{0\leq m<K} \left\langle Q[l][m] = \sum^{0\leq k<K} \left\langle \sum^{\substack{0\leq i<K \\ \alpha=F[l][i]}} \left\langle \begin{cases} c[k] & \text{if } i = m \\ d[k][\alpha] & \text{otherwise} \end{cases} \right\rangle \right\rangle \right\rangle \right\rangle \tag{4.90}$$

The polynomial coefficients follow by Cramer's rule if the determinant of the base matrix of the corresponding polynomial is non-zero.

$$\sum^{0\leq l<L} \left\langle \text{if}\,(\det(G[l]) \neq 0)\,\text{then} \left(\sum^{\substack{0\leq i<K \\ \alpha=A[i]}} \left\langle d[l][\alpha] = \frac{Q[l][i]}{G[l]} \right\rangle \right) \right\rangle \tag{4.91}$$

The determinant of the maximum absolute value (2.36) is determined.

$$\sum^{0\leq l<L} \left\langle \; H[l] = \det(G[l]) \; \right\rangle ; \qquad\qquad T = \mathrm{iAbsMax}(H) \tag{4.92}$$

A polynomial does not exist if the determinant of its base matrix equals zero. Therefore the polynomial of the largest determinant is selected as solution. Other criteria may apply.

$$\text{if}\,(\det(G[T]) \neq 0)\,\text{then} \left(f(\vec{x}) = g[J](\vec{x})\,; \qquad \sum^{\alpha\in E[T]} \langle a[\alpha] = d[T][\alpha] \rangle \right) \tag{4.93}$$

4.5 Example of Two-Dimensional von-Neumann Conditions

4.5.1 Four Dirichlet Conditions

The terms of a polynomial of a two-dimensional position are determined.

$$\begin{aligned} f(\vec{x}) = a[\langle 0;0\rangle] + a[\langle 0;1\rangle] * \vec{x}^{\langle 0;1\rangle} + a[\langle 1;0\rangle] * \vec{x}^{\langle 1;0\rangle} \\ + a[\langle 0;2\rangle] * \vec{x}^{\langle 0;2\rangle} + a[\langle 1;1\rangle] * \vec{x}^{\langle 1;1\rangle} + a[\langle 2;0\rangle] * \vec{x}^{\langle 2;0\rangle} \end{aligned} \tag{4.94}$$

Two von-Neumann conditions are determined at the origin by two orthogonal distances.

$$\vec{V}[0] * a[\langle 1;0\rangle] + \vec{V}[1] * a[\langle 0;1\rangle] = b[0] \tag{4.95}$$

$$-\vec{V}[1] * a[\langle 1;0\rangle] + \vec{V}[0] * a[\langle 0;1\rangle] = b[1] \tag{4.96}$$

The von-Neumann conditions determine the first order coefficients. The denominator equals One if the magnitude of the distances equals One.

$$a[\langle 1;0\rangle] = \frac{\det\left(\begin{bmatrix} b[0] & \vec{V}[1] \\ b[1] & \vec{V}[0] \end{bmatrix}\right)}{\det\left(\begin{bmatrix} \vec{V}[0] & \vec{V}[1] \\ -\vec{V}[1] & \vec{V}[0] \end{bmatrix}\right)} = \frac{b[0] * \vec{V}[0] - b[1] * \vec{V}[1]}{\vec{V}[0]^2 + \vec{V}[1]^2} \tag{4.97a}$$

$$a[\langle 0;1\rangle] = \frac{\det\left(\begin{bmatrix} \vec{V}[0] & b[0] \\ -\vec{V}[1] & b[1] \end{bmatrix}\right)}{\det\left(\begin{bmatrix} \vec{V}[0] & \vec{V}[1] \\ -\vec{V}[1] & \vec{V}[0] \end{bmatrix}\right)} = \frac{b[1] * \vec{V}[0] + b[0] * \vec{V}[1]}{\vec{V}[0]^2 + \vec{V}[1]^2} \tag{4.97b}$$

The remaining coefficients are determined by the Dirichlet conditions.

$$y[i] - a[\langle 0;1\rangle] * \vec{x}[i]^{\langle 0;1\rangle} - a[\langle 1;0\rangle] * \vec{x}[i]^{\langle 1;0\rangle} = c[i] \tag{4.98}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \vec{U}[1]^{\langle 0;2\rangle} & \vec{U}[1]^{\langle 1;1\rangle} & \vec{U}[1]^{\langle 2;0\rangle} \\ 1 & \vec{U}[2]^{\langle 0;2\rangle} & \vec{U}[2]^{\langle 1;1\rangle} & \vec{U}[2]^{\langle 2;0\rangle} \\ 1 & \vec{U}[3]^{\langle 0;2\rangle} & \vec{U}[3]^{\langle 1;1\rangle} & \vec{U}[3]^{\langle 2;0\rangle} \end{bmatrix} * \begin{bmatrix} a[\langle 0;0\rangle] \\ a[\langle 0;2\rangle] \\ a[\langle 1;1\rangle] \\ a[\langle 2;0\rangle] \end{bmatrix} = \begin{bmatrix} y[0] \\ c[1] \\ c[2] \\ c[3] \end{bmatrix} \tag{4.99}$$

The determinant of the base matrix equals the product of the three lines through the origin and one other point.

$$\det(G) = \begin{aligned} &\left(\vec{U}[1]^{\langle 1;0\rangle} * \vec{U}[2]^{\langle 0;1\rangle} - \vec{U}[1]^{\langle 0;1\rangle} * \vec{U}[2]^{\langle 1;0\rangle}\right) \\ &* \left(\vec{U}[2]^{\langle 1;0\rangle} * \vec{U}[3]^{\langle 0;1\rangle} - \vec{U}[2]^{\langle 0;1\rangle} * \vec{U}[3]^{\langle 1;0\rangle}\right) \\ &* \left(\vec{U}[3]^{\langle 1;0\rangle} * \vec{U}[1]^{\langle 0;1\rangle} - \vec{U}[3]^{\langle 0;1\rangle} * \vec{U}[1]^{\langle 1;0\rangle}\right) \end{aligned} \tag{4.100}$$

The determinant is Zero if any two points are coincident or any two lines are collinear. Therefore the operator is not determined if the origin and any two points lie on a linear boundary.

4.5.2 Five Dirichlet Conditions

A polynomial of a two-dimensional position is determined by seven terms.

$$f(\vec{x}) = a[\langle 0;1\rangle] * \vec{x}^{\langle 0;1\rangle} + a[\langle 1;0\rangle] * \vec{x}^{\langle 1;0\rangle} \tag{4.101a}$$

$$g(\vec{x}) = a[\langle 0;0\rangle] + a[\langle 0;2\rangle] * \vec{x}^{\langle 0;2\rangle} + a[\langle 1;1\rangle] * \vec{x}^{\langle 1;1\rangle} + a[\langle 2;0\rangle] * \vec{x}^{\langle 2;0\rangle} \tag{4.101b}$$

$$h[\alpha](\vec{x}) = f(\vec{x}) + g(\vec{x}) + a[\alpha] * \vec{x}^{\alpha} \tag{4.101c}$$

Two von-Neumann conditions determine the first order coefficients according to (4.97).

$$\vec{V} = \overrightarrow{\langle 1;0\rangle}; \qquad a[\langle 1;0\rangle] = b[0]; \qquad a[\langle 0;1\rangle] = b[1] \tag{4.102}$$

The remaining coefficients are determined by the Dirichlet conditions.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & \vec{U}[1]^{\langle 0;2\rangle} & \vec{U}[1]^{\langle 1;1\rangle} & \vec{U}[1]^{\langle 2;0\rangle} & \vec{U}[1]^{\alpha} \\ 1 & \vec{U}[2]^{\langle 0;2\rangle} & \vec{U}[2]^{\langle 1;1\rangle} & \vec{U}[2]^{\langle 2;0\rangle} & \vec{U}[2]^{\alpha} \\ 1 & \vec{U}[3]^{\langle 0;2\rangle} & \vec{U}[3]^{\langle 1;1\rangle} & \vec{U}[3]^{\langle 2;0\rangle} & \vec{U}[3]^{\alpha} \\ 1 & \vec{U}[4]^{\langle 0;2\rangle} & \vec{U}[4]^{\langle 1;1\rangle} & \vec{U}[4]^{\langle 2;0\rangle} & \vec{U}[4]^{\alpha} \end{bmatrix} * \begin{bmatrix} a[\langle 0;0\rangle] \\ a[\langle 0;2\rangle] \\ a[\langle 1;1\rangle] \\ a[\langle 2;0\rangle] \\ a[\alpha] \end{bmatrix} = \begin{bmatrix} y[0] - f\left(\vec{\mathbf{0}}\right) \\ y[1] - f\left(\vec{U}[1]\right) \\ y[2] - f\left(\vec{U}[2]\right) \\ y[3] - f\left(\vec{U}[3]\right) \\ y[4] - f\left(\vec{U}[4]\right) \end{bmatrix} \tag{4.103}$$

Four points span a trapezoid along the distance $\vec{V}$.

$$\vec{U}[1] = \overrightarrow{\langle 2*d;0\rangle} \qquad \vec{U}[2] = \overrightarrow{\langle d;2*d\rangle} \qquad \vec{U}[3] = \overrightarrow{\langle -d;2*d\rangle} \qquad \vec{U}[4] = \overrightarrow{\langle -2*d;0\rangle} \tag{4.104}$$

The points are substituted into the system of equations.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 4*d^2 & \vec{U}[1]^{\alpha} \\ 1 & 4*d^2 & 2*d^2 & d^2 & \vec{U}[2]^{\alpha} \\ 1 & 4*d^2 & -2*d^2 & d^2 & \vec{U}[3]^{\alpha} \\ 1 & 0 & 0 & 4*d^2 & \vec{U}[4]^{\alpha} \end{bmatrix} * \begin{bmatrix} a[\langle 0;0\rangle] \\ a[\langle 0;2\rangle] \\ a[\langle 1;1\rangle] \\ a[\langle 2;0\rangle] \\ a[\alpha] \end{bmatrix} = \begin{bmatrix} y[0] - f\left(\vec{\mathbf{0}}\right) \\ y[1] - f\left(\vec{U}[1]\right) \\ y[2] - f\left(\vec{U}[2]\right) \\ y[3] - f\left(\vec{U}[3]\right) \\ y[4] - f\left(\vec{U}[4]\right) \end{bmatrix} \tag{4.105}$$

The determinant of the base matrix is given.

$$\det(G[\alpha]) = 64 * d^6 * \left(\vec{U}[1]^{\alpha} - \vec{U}[4]^{\alpha}\right) \tag{4.106}$$

Four tuples of two dimensions and a horizontal check sum of Three exist.

$$\alpha \in (\quad \langle 0;3\rangle ; \langle 1;2\rangle ; \langle 2;1\rangle ; \langle 3;0\rangle \quad) \tag{4.107}$$

The determinants are evaluated.

$$\det(G[\langle 0;3\rangle]) = \det(G[\langle 1;2\rangle]) = \det(G[\langle 2;1\rangle]) = 0; \qquad \det(G[\langle 3;0\rangle]) = 1024 * d^9 \tag{4.108}$$

Only one polynomial fulfills the conditions.

$$h[\langle 3;0\rangle](\vec{x}) = f(\vec{x}) + g(\vec{x}) + a[\langle 3;0\rangle] * \vec{x}^{\langle 3;0\rangle} \tag{4.109}$$

Chapter 5

Integration Method $\int$

The Taylor series k of polynomial f is determined.

$$k(\vec{u};\vec{v}) = \sum^{\beta\in B}\left\{\frac{\vec{v}^{\beta}}{\beta!_*} * \partial f[\beta](\vec{u})\right\}; \qquad \begin{array}{l} A = \mathbb{T}\text{nss}\,(n; s) \\ B = \mathbb{T}\text{nss}\,(n; \mathcal{G}\,(A)) \end{array} \tag{5.1}$$

The integration method $\int(f;\Psi)$ results a polynomial g of which the derivative of order Ψ equals polynomial f. Therefore the Taylor coefficients are increased by Ψ. Furthermore an integration variable is added which is determined by (5.13).

$$h(\vec{u};\vec{v}) = c + \sum^{\beta\in B}\left\{\frac{\vec{v}^{\beta+\Psi}}{(\beta+\Psi)!_*} * \partial f[\beta](\vec{u})\right\} \tag{5.2}$$

The derivatives are given explicitly according to (4.7).

$$h(\vec{u};\vec{v}) = c + \sum^{\beta\in B}\left\{\frac{\vec{v}^{\beta+\Psi}}{(\beta+\Psi)!_*} * \sum_{\alpha\geq\beta}^{\alpha\in A}\left\{a[\alpha] * (\alpha¡\beta)_* * \vec{u}^{\alpha-\beta}\right\}\right\} \tag{5.3}$$

The sums are transposed.

$$h(\vec{u};\vec{v}) = c + \sum^{\alpha\in A}\left\{a[\alpha] * \sum_{\beta\leq\alpha}^{\beta\in B}\left\{\frac{\vec{v}^{\beta+\Psi}}{(\beta+\Psi)!_*} * (\alpha¡\beta)_* * \vec{u}^{\alpha-\beta}\right\}\right\} \tag{5.4}$$

The polynomial is to depend on position $\vec{x}$ which may be separated into $\vec{u}$ and $\vec{v}$. This separation, however, cannot be reversed due to the increase of the Taylor coefficients. Thus the integration is developed at a single constant location $\vec{U}$ which gives the polynomial g immediately.

$$\vec{x} = \vec{u} + \vec{v}; \qquad \vec{u} = \vec{U} = \text{const}; \qquad \vec{v} = \vec{x} - \vec{U} \tag{5.5}$$

$$h\left(\vec{U}; \vec{x} - \vec{U}\right) = c + \sum^{\alpha\in A}\left\{a[\alpha] * \sum_{\beta\leq\alpha}^{\beta\in B}\left\{\frac{(\alpha¡\beta)_*}{(\beta+\Psi)!_*} * \vec{U}^{\alpha-\beta} * \left(\vec{x} - \vec{U}\right)^{\beta+\Psi}\right\}\right\} = g(\vec{x}) \tag{5.6}$$

The binomial expansion is applied by $\vec{x}$.

$$C = \mathbb{T}\text{nss}\,(n; s + \mathcal{S}(\Psi)) \tag{5.7}$$

$$g(\vec{x}) = c + \sum^{\alpha\in A}\left\{a[\alpha]*\sum_{\beta\leq\alpha}^{\beta\in B}\left\{\frac{(\alpha¡\beta)_*}{(\beta+\Psi)!_*}*\vec{U}^{\alpha-\beta}*\sum_{\gamma\leq\beta+\Psi}^{\gamma\in C}\left\{\binom{\beta+\Psi}{\gamma}_* * \vec{x}^{\gamma} * \overrightarrow{-U}^{\beta+\Psi-\gamma}\right\}\right\}\right\} \tag{5.8}$$

The inner sums are transposed in order to group all constants.

$$g(\vec{x}) = c + \sum^{\alpha\in A}\left\{a[\alpha]*\sum_{\gamma\leq\alpha+\Psi}^{\gamma\in C}\left\{\vec{x}^{\gamma}*\sum_{\substack{\beta\leq\alpha\\ \beta+\Psi\geq\gamma}}^{\beta\in B}\left\{\frac{(\alpha¡\beta)_*}{(\beta+\Psi)!_*}*\vec{U}^{\alpha-\beta}*\binom{\beta+\Psi}{\gamma}_* * \overrightarrow{-U}^{\beta+\Psi-\gamma}\right\}\right\}\right\} \tag{5.9}$$

The expression of constants is discussed.

$$\begin{aligned} t(\alpha;\beta;\gamma) &= \frac{(\alpha¡\beta)_*}{(\beta+\Psi)!_*}*\vec{U}^{\alpha-\beta}*\binom{\beta+\Psi}{\gamma}*\overrightarrow{-U}^{\beta+\Psi-\gamma} \\ &= (-1)^{\beta+\Psi-\gamma}*\frac{(\alpha¡\beta)_*}{(\beta+\Psi)!_*}*\binom{\beta+\Psi}{\gamma}_* * \vec{U}^{\alpha+\Psi-\gamma} \end{aligned} \tag{5.10a}$$

The binomial coefficient and faculties are rearranged by (2.60a) such that only one factor depends on β.

$$\frac{(\alpha¡\beta)_*}{(\beta+\Psi)!_*}*\binom{\beta+\Psi}{\gamma}_* = \frac{(\ (\alpha+\Psi)\,¡\,(\beta+\Psi)\ )_*}{(\ (\alpha+\Psi)\,¡\Psi\ )_* * (\beta+\Psi)!_*}*\binom{\beta+\Psi}{\gamma}_* \tag{5.10b}$$

$$= \frac{1}{(\ (\alpha+\Psi)\,¡\Psi\ )_*}*\binom{\alpha+\Psi}{\beta+\Psi}_* * \binom{\beta+\Psi}{\gamma}_* \tag{5.10c}$$

$$= \frac{1}{(\ (\alpha+\Psi)\,¡\Psi\ )_*}*\binom{\alpha+\Psi}{\gamma}_* * \binom{\alpha+\Psi-\gamma}{\beta+\Psi-\gamma}_* \tag{5.10d}$$

The expression of constants is noted by a product of two methods.

$$p(\alpha;\gamma) = \frac{\vec{U}^{\alpha+\Psi-\gamma}}{((\alpha+\Psi)\,¡\Psi)_*}*\binom{\alpha+\Psi}{\gamma} \tag{5.10e}$$

$$q(\alpha;\beta;\gamma) = (-1)^{\beta+\Psi-\gamma}*\binom{\alpha+\Psi-\gamma}{\beta+\Psi-\gamma} \tag{5.10f}$$

$$p(\alpha;\gamma)*q(\alpha;\beta;\gamma) = t(\alpha;\beta;\gamma) \tag{5.10g}$$

The product is substituted into (5.9). The value of the innermost sum maps to a part or an entire line of Pascal's triangle with alternating signs.

$$g(\vec{x}) = c + \sum^{\alpha\in A}\left\{a[\alpha]*\sum_{\gamma\leq\alpha+\Psi}^{\gamma\in C}\left\{p(\alpha;\gamma)*\vec{x}^{\gamma}*\sum_{\substack{\beta\leq\alpha\\ \beta+\Psi\geq\gamma}}^{\beta\in B}\{q(\alpha;\beta;\gamma)\}\right\}\right\} \tag{5.10h}$$

The zeroth line of Pascal's triangle follows under two conditions.

$$\text{if}\,((\alpha=\beta)*(\alpha+\Psi=\gamma))\,\text{then}\left(p(\alpha;\alpha+\Psi) = \frac{1}{((\alpha+\Psi)\,¡\Psi)_*};\quad q(\alpha;\beta;\alpha+\Psi)=1\right) \tag{5.11a}$$

This expression results once for each coefficient α and is noted explicitly. Therefore the condition on the sum over C is modified.

$$g(\vec{x}) = c + \sum^{\alpha\in A} \left\{ a[\alpha] * \sum_{\gamma<\alpha+\Psi}^{\gamma\in C} \left\{ p(\alpha;\gamma) * \vec{x}^{\gamma} * \sum_{\substack{\beta\leq\alpha\\ \beta+\Psi\geq\gamma}}^{\beta\in B} \{q(\alpha;\beta;\gamma)\} \right\} \right\} + \sum^{\alpha\in A} \left\{ \frac{a[\alpha] * \vec{x}^{\alpha+\Psi}}{((\alpha+\Psi)\,¡\Psi)_*} \right\} \quad (5.11b)$$

The value of the innermost sum cancels if it maps to a full line of Pascal's triangle with alternating signs.

$$\text{if}\,((\gamma < \alpha + \Psi) * (\gamma[i] \geq \Psi[i]))\,\text{then} \left(\sum_{\substack{\beta\leq\alpha\\ \beta+\Psi\geq\gamma}}^{\beta\in B} \{q(\alpha;\beta;\gamma)\} = 0 \right) \quad (5.12a)$$

Therefore the condition on the sum over C is modified. The point of integration $\vec{U}$ is significant only in case $\gamma < \Psi$. The integration polynomial is determined.

$$g(\vec{x}) = c + \sum^{\alpha\in A} \left\{ a[\alpha] * \sum_{\gamma<\Psi}^{\gamma\in C} \left\{ p(\alpha;\gamma) * \vec{x}^{\gamma} * \sum_{\substack{\beta\leq\alpha\\ \beta+\Psi\geq\gamma}}^{\beta\in B} \{q(\alpha;\beta;\gamma)\} \right\} \right\} + \sum^{\alpha\in A} \left\{ \frac{a[\alpha] * \vec{x}^{\alpha+\Psi}}{((\alpha+\Psi)\,¡\Psi)_*} \right\} \quad (5.12b)$$

$$g(\vec{x}) = c + \textstyle\int(f;\Psi) \quad (5.12c)$$

The integration polynomial equals the integrand if the increase of the Talyor coefficients Ψ equals Zero. The integration variable c may be any polynomial of $\vec{x}$ with a degree less than that of increase Ψ.

$$c(\vec{x}) = \sum_{\delta<\Psi}^{\delta\in D} \left\{ b[\delta] * \vec{x}^{\delta} \right\}; \qquad D = \mathbb{T}\text{nss}\,(n; \mathcal{S}(\Psi)) \quad (5.13)$$

The integration polynomial simplifies in case the point of integration equals the origin.

$$\text{if}\left(\vec{U} = \vec{\mathbf{0}}\right)\text{then}\left(g(\vec{x}) = c + \sum^{\alpha\in A} \left\{ \frac{a[\alpha] * \vec{x}^{\alpha+\Psi}}{((\alpha+\Psi)\,¡\Psi)_*} \right\} \right) \quad (5.14)$$

Other integration methods may be defined by integer operations.

5.1 Simple One-Dimensional Integration

The Taylor coefficients of a Taylor series of one dimension are increased by One. Therefore array B equals array A.

$$h(\vec{u};\vec{v}) = c + \sum^{\beta\in B} \left\{ \frac{\vec{v}^{\beta+1}}{(\beta+1)!_*} * \partial f[\beta](\vec{u}) \right\}; \qquad \begin{array}{l} A = \mathbb{T}\text{nss}\,(1; s) \\ B = \mathbb{T}\text{nss}\,(1; \mathcal{G}\,(A)) = A \end{array} \quad (5.15)$$

The sum over C of (5.12b) reduces to one element of $\vec{x}^0 = 1$ since $\gamma < \Psi$. The second condition on the innermost sum is fulfilled since $\beta + 1 \geq 0$.

$$g(\vec{x}) = c + \sum^{\alpha \in A} \left\{ a[\alpha] * p(\alpha; 0) * \sum_{\beta \leq \alpha}^{\beta \in A} \{q(\alpha; \beta; 0)\} \right\} + \sum^{\alpha \in A} \left\{ \frac{a[\alpha] * \vec{x}^{\alpha+1}}{((\alpha+1)\,¡1)_*} \right\} \tag{5.16}$$

The sum of β maps to a full line but the zeroth element of Pascal's triangle with alternating signs.

$$\sum_{\beta \leq \alpha}^{\beta \in A} \{q(\alpha; \beta; 0)\} = \sum_{\beta \leq \alpha}^{\beta \in A} \left\{ (-1)^{\beta+1} * \binom{\alpha+1}{\beta+1} \right\} = -1 \tag{5.17}$$

Method p simplifies.

$$p(\alpha; 0) = \frac{\vec{U}^{\alpha+1}}{((\alpha+1)\,¡1)_*} * \binom{\alpha+1}{0} = \frac{\vec{U}^{\alpha+1}}{\alpha+1} \tag{5.18}$$

The simple one-dimensional integration polynomial is determined. The integration variable c is constant.

$$g(\vec{x}) = c - \sum^{\alpha \in A} \left\{ \frac{a[\alpha] * \vec{U}^{\alpha+1}}{\alpha+1} \right\} + \sum^{\alpha \in A} \left\{ \frac{a[\alpha] * \vec{x}^{\alpha+1}}{\alpha+1} \right\}; \qquad c = \text{const} \tag{5.19}$$

5.2 Example of Two Dimensions

A polynomial of two terms of a two-dimensional position is given.

$$f(\vec{x}) = \sum^{\alpha \in A} \{a[\alpha] * \vec{x}^{\alpha}\}; \qquad A = (\langle 1; 1\rangle; \langle 2; 0\rangle) \tag{5.20}$$

A polynomial g is to be determined of which the derivative of order $\langle 2; 1\rangle$ equals the polynomial f.

$$h(\vec{u}; \vec{v}) = c + \sum^{\beta \in B} \left\{ \frac{\vec{v}^{\beta + \langle 2;1\rangle}}{(\beta + \langle 2; 1\rangle)!_*} * \partial f[\beta](\vec{u}) \right\}; \quad B = (\langle 0; 0\rangle; \langle 0; 1\rangle; \langle 1; 0\rangle; \langle 1; 1\rangle; \langle 2; 0\rangle) \tag{5.21}$$

The integration is developed at the constant position $\vec{U}$.

$$h\left(\vec{U}; \vec{x} - \vec{U}\right) = c + \sum^{\beta \in B} \left\{ \frac{\left(\vec{x} - \vec{U}\right)^{\beta + \langle 2;1\rangle}}{(\beta + \langle 2; 1\rangle)!_*} * \partial f[\beta]\left(\vec{U}\right) \right\} = g(\vec{x}); \qquad \vec{U} = \text{const} \tag{5.22}$$

The integration is expressed according to (5.10h).

$$C = \begin{pmatrix} \langle 0; 0\rangle; \langle 0; 1\rangle; \langle 1; 0\rangle\, \langle 0; 2\rangle; \langle 1; 1\rangle; \langle 2; 0\rangle; \langle 1; 2\rangle; \langle 2; 1\rangle; \langle 3; 0\rangle; \\ \langle 2; 2\rangle; \langle 3; 1\rangle; \langle 4; 0\rangle; \langle 3; 2\rangle; \langle 4; 1\rangle \end{pmatrix} \tag{5.23}$$

$$g(\vec{x}) = c + \sum^{\alpha \in A} \left\{ a[\alpha] * \sum_{\gamma \leq \alpha + \langle 2;1\rangle}^{\gamma \in C} \left\{ p(\alpha; \gamma) * \vec{x}^{\gamma} * \sum_{\substack{\beta \leq \alpha \\ \beta + \langle 2;1\rangle \geq \gamma}}^{\beta \in B} \{q(\alpha; \beta; \gamma)\} \right\} \right\} \tag{5.24}$$

All terms of α equal $\langle 1;1\rangle$ are determined.

$$\alpha = \langle 1;1\rangle\,; \qquad \gamma = \langle 0;0\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{12}\vec{U}^{\langle 3;2\rangle}; \qquad \sum^{\beta} = 3 \tag{5.25a}$$

$$\alpha = \langle 1;1\rangle\,; \qquad \gamma = \langle 0;1\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{6}\vec{U}^{\langle 3;1\rangle}; \qquad \sum^{\beta} = 0 \tag{5.25b}$$

$$\alpha = \langle 1;1\rangle\,; \qquad \gamma = \langle 1;0\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{4}\vec{U}^{\langle 2;2\rangle}; \qquad \sum^{\beta} = 1 \tag{5.25c}$$

$$\alpha = \langle 1;1\rangle\,; \qquad \gamma = \langle 0;2\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{12}\vec{U}^{\langle 3;0\rangle}; \qquad \sum^{\beta} = 2 \tag{5.25d}$$

$$\alpha = \langle 1;1\rangle\,; \qquad \gamma = \langle 1;1\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{2}\vec{U}^{\langle 2;1\rangle}; \qquad \sum^{\beta} = 0 \tag{5.25e}$$

$$\alpha = \langle 1;1\rangle\,; \qquad \gamma = \langle 2;0\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{4}\vec{U}^{\langle 1;2\rangle}; \qquad \sum^{\beta} = 0 \tag{5.25f}$$

$$\alpha = \langle 1;1\rangle\,; \qquad \gamma = \langle 1;2\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{4}\vec{U}^{\langle 2;0\rangle}; \qquad \sum^{\beta} = 0 \tag{5.25g}$$

$$\alpha = \langle 1;1\rangle\,; \qquad \gamma = \langle 2;1\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{2}\vec{U}^{\langle 1;1\rangle}; \qquad \sum^{\beta} = 0 \tag{5.25h}$$

$$\alpha = \langle 1;1\rangle\,; \qquad \gamma = \langle 3;0\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{12}\vec{U}^{\langle 0;2\rangle}; \qquad \sum^{\beta} = -1 \tag{5.25i}$$

$$\alpha = \langle 1;1\rangle\,; \qquad \gamma = \langle 2;2\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{3}\vec{U}^{\langle 0;1\rangle}; \qquad \sum^{\beta} = 0 \tag{5.25j}$$

$$\alpha = \langle 1;1\rangle\,; \qquad \gamma = \langle 3;1\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{6}\vec{U}^{\langle 0;1\rangle}; \qquad \sum^{\beta} = 0 \tag{5.25k}$$

$$\alpha = \langle 1;1\rangle\,; \qquad \gamma = \langle 3;2\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{12}\vec{U}^{\langle 0;0\rangle}; \qquad \sum^{\beta} = 1 \tag{5.25l}$$

All terms of α equal $\langle 2;0\rangle$ are determined.

$$\alpha = \langle 2;0\rangle\,; \qquad \gamma = \langle 0;0\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{12}\vec{U}^{\langle 4;1\rangle}; \qquad \sum^{\beta} = -3 \tag{5.26a}$$

$$\alpha = \langle 2;0\rangle\,; \qquad \gamma = \langle 0;1\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{12}\vec{U}^{\langle 4;0\rangle}; \qquad \sum^{\beta} = 3 \tag{5.26b}$$

$$\alpha = \langle 2;0\rangle\,; \qquad \gamma = \langle 1;0\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{3}\vec{U}^{\langle 3;1\rangle}; \qquad \sum^{\beta} = 1 \tag{5.26c}$$

$$\alpha = \langle 2;0\rangle\,; \qquad \gamma = \langle 1;1\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{3}\vec{U}^{\langle 3;0\rangle}; \qquad \sum^{\beta} = -1 \tag{5.26d}$$

$$\alpha = \langle 2;0\rangle\,; \qquad \gamma = \langle 2;0\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{2}\vec{U}^{\langle 2;1\rangle}; \qquad \sum^{\beta} = 0 \tag{5.26e}$$

$$\alpha = \langle 2;0\rangle\,; \qquad \gamma = \langle 2;1\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{2}\vec{U}^{\langle 2;0\rangle}; \qquad \sum^{\beta} = 0 \tag{5.26f}$$

$$\alpha = \langle 2;0\rangle\,; \qquad \gamma = \langle 3;0\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{3}\vec{U}^{\langle 1;1\rangle}; \qquad \sum^{\beta} = 0 \tag{5.26g}$$

$$\alpha = \langle 2;0\rangle\,; \qquad \gamma = \langle 3;1\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{3}\vec{U}^{\langle 1;0\rangle}; \qquad \sum^{\beta} = 0 \tag{5.26h}$$

$$\alpha = \langle 2;0\rangle\,; \qquad \gamma = \langle 4;0\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{12}\vec{U}^{\langle 0;1\rangle}; \qquad \sum^{\beta} = -1 \tag{5.26i}$$

$$\alpha = \langle 2;0\rangle\,; \qquad \gamma = \langle 4;1\rangle\,; \qquad p(\alpha;\gamma) = \frac{1}{12}\vec{U}^{\langle 0;0\rangle}; \qquad \sum^{\beta} = 1 \tag{5.26j}$$

The integration polynomial is determined.

$$\begin{aligned} g(\vec{x}) = c &+ \frac{1}{4} * a[\langle 1;1\rangle] * \vec{U}^{\langle 3;2\rangle} + \frac{1}{4} * a[\langle 1;1\rangle] * \vec{U}^{\langle 2;2\rangle} * \vec{x}^{\langle 1;0\rangle} + \frac{1}{6} * a[\langle 1;1\rangle] * \vec{U}^{\langle 3;0\rangle} * \vec{x}^{\langle 0;2\rangle} \\ &- \frac{1}{12} * a[\langle 1;1\rangle] * \vec{U}^{\langle 0;2\rangle} * \vec{x}^{\langle 3;0\rangle} + \frac{1}{12} * a[\langle 1;1\rangle] * \vec{x}^{\langle 3;2\rangle} \\ &- \frac{1}{4} * a[\langle 2;0\rangle] * \vec{U}^{\langle 4;1\rangle} + \frac{1}{4} * a[\langle 2;0\rangle] * \vec{U}^{\langle 4;0\rangle} * \vec{x}^{\langle 0;1\rangle} \\ &+ \frac{1}{3} * a[\langle 2;0\rangle] * \vec{U}^{\langle 3;1\rangle} * \vec{x}^{\langle 1;0\rangle} - \frac{1}{3} * a[\langle 2;0\rangle] * \vec{U}^{\langle 3;0\rangle} * \vec{x}^{\langle 1;1\rangle} \\ &- \frac{1}{12} * a[\langle 2;0\rangle] * \vec{U}^{\langle 0;1\rangle} * \vec{x}^{\langle 4;0\rangle} + \frac{1}{12} * a[\langle 2;0\rangle]\, \vec{x}^{\langle 4;1\rangle} \end{aligned}$$

The solution reduces if the integration is developed at the origin.

$$\vec{U} = \vec{\mathbf{0}}; \qquad g(\vec{x}) = c + \frac{1}{12} * a[\langle 1;1\rangle] * \vec{x}^{\langle 3;2\rangle} + \frac{1}{12} * a[\langle 2;0\rangle]\, \vec{x}^{\langle 4;1\rangle}$$

The integration variable is a polynomial of which the coefficients may equal Zero.

$$c = b[\langle 0;0\rangle] * \vec{x}^{\langle 0;0\rangle} + b[\langle 1;0\rangle] * \vec{x}^{\langle 1;0\rangle} \tag{5.27}$$

Chapter 6

Transposed Polynomials

The transposed form expresses a polynomial in terms of base polynomials w. The polynomial is determined by differential expressions of a position $\vec{u}$ in accordance to the differential method, page 27. These expressions are given in terms of constant points. Therefore a polynomial of any other position such as $\vec{x}$ is determined consistently by these conditions.

A polynomial and its significant derivatives are determined.

$$f(\vec{x}) = \sum^{\alpha\in A} \{a[\alpha] * \vec{x}^{\alpha}\} \qquad A \in \mathbb{T}\mathrm{n}(n) \tag{6.1}$$

$$\sum^{\beta\in B} \langle \partial f[\beta](\vec{u}) = \partial\left(\beta; f(\vec{x}); \vec{u}\right)\rangle \qquad B = \mathbb{T}\mathrm{nss}\left(n; \mathcal{G}\left(A\right)\right) \tag{6.2}$$

An array of unique constant positions is determined.

$$\sum^{0\le j<J} \left\langle \vec{U}[j] = \mathrm{const} \right\rangle \tag{6.3}$$

A differential term is a product of a constant and a derivative. A differential equation is a sum of differential terms. A complete differential equation combines all derivatives at all locations. The constant of a term may equal Zero such that a term is insignificant. An array of complete differential equations on an array of unique locations is determined.

$$\sum^{0\le k<K} \left\langle \sum^{0\le j<J} \left\{ \sum^{\beta\in B} \left\{ b[k][j][\beta] * \partial f[\beta]\left(\vec{U}[j]\right) \right\} \right\} = c[k] \right\rangle; \quad b[k][j][\beta] \in \mathbb{R};\ c[k] \in \mathbb{R} \tag{6.4}$$

The K differential equations determine K coefficients of a polynomial if such a polynomial exists. A polynomial requires an array of tuples. The largest possible degree of a complete array with less than K tuples (2.69c) is determined.

$$p = \mathrm{gSnt}\left(n; K\right) \tag{6.5}$$

The complete array of tuples of a horizontal check sum less than or equal to p is determined.

$$C = \mathbb{T}\mathrm{nss}\left(n; p\right) \tag{6.6}$$

The complete array of tuples of a horizontal check sum of $(p+1)$ is determined. This array holds at least one element.

$$D = \mathbb{CT}\mathrm{n}\left(\mathbb{T}\mathrm{ns}\,(n; p+1)\,; K - \binom{n+p}{p}\right); \qquad L = \mathrm{size}\,(D)\,;\ L > 0 \tag{6.7}$$

L arrays of K tuples (2.73c) are determined that may determine L polynomials.

$$\sum^{0\leq l<L} \left\langle\ E[l] = \langle C; D[l]\rangle\ \right\rangle \tag{6.8}$$

The degrees of all arrays are equal and the complete array of tuples of this degree equals B of (4.73).

$$\sum^{0\leq l<L} \left\langle\ \mathbb{T}\mathrm{nss}\,(n; \mathcal{G}\,(E[l])) = B\ \right\rangle \tag{6.9}$$

The possible polynomials are given.

$$\sum^{0\leq l<L} \left\langle g[l](\vec{x}) = \sum^{\alpha\in E[l]} \{d[l][\alpha] * \vec{x}^{\alpha}\} \right\rangle \tag{6.10}$$

A system of differential equations is determined with K equations for each of L polynomials.

$$\sum^{0\leq l<L} \left\langle \sum^{0\leq k<K} \left\langle \sum^{0\leq j<J} \left\{ \sum^{\beta\in B} \left\{ b[k][j][\beta] * \partial g[l][\beta]\left(\vec{U}[j]\right)\right\} \right\} = c[k] \right\rangle \right\rangle \tag{6.11}$$

Each equation is multiplied by a base polynomial or weight w. A sum of all weighted equation is determined. The sum of weights w and constants c is the transposed polynomial that equals the polynomial g.

$$\sum^{0\leq l<L} \left\langle \begin{array}{rcl} g[l](\vec{x}) & = & \displaystyle\sum^{0\leq k<K} \left\{ w[l][k](\vec{x}) * \sum^{0\leq j<J} \left\{ \sum^{\beta\in B} \left\{ b[k][j][\beta] * \partial g[l][\beta]\left(\vec{U}[j]\right)\right\}\right\}\right\} \\ & = & \displaystyle\sum^{0\leq k<K} \{w[l][k](\vec{x}) * c[k]\} \end{array} \right\rangle \tag{6.12}$$

The canonical form in terms of polynomial coefficients is required. Therefore the derivatives are noted explicitly.

$$\sum^{0\leq l<L} \left\langle g[l](\vec{x}) = \sum^{0\leq k<K} \left\{ w[l][k](\vec{x}) * \sum^{0\leq j<J} \left\{ \sum^{\beta\in B} \left\{ b[k][j][\beta] * \sum_{\alpha\geq\beta}^{\alpha\in E[l]} \left\{ a[\alpha] * (\alpha¡\beta)_* * \vec{U}[j]^{\alpha-\beta} \right\}\right\}\right\}\right\} \right\rangle \tag{6.13}$$

The innermost sum is put in front such that the terms are grouped by the polynomial coefficients.

$$\sum^{0\leq l<L} \left\langle g[l](\vec{x}) = \sum^{\alpha\in E[l]} \left\{ a[\alpha] * \sum^{0\leq k<K} \left\{ w[l][k](\vec{x}) * \sum^{0\leq j<J} \left\{ \sum_{\beta\leq\alpha}^{\beta\in B} \left\{ b[k][j][\beta] * (\alpha¡\beta)_* * \vec{U}[j]^{\alpha-\beta} \right\}\right\}\right\}\right\} \right\rangle \tag{6.14}$$

The value of the inner double sum is constant and determined by a method.

$$H(\alpha;k) = \sum^{0\le j<J} \left\{ \sum_{\beta\le\alpha}^{\beta\in B} \left\{ b[k][j][\beta] * (\alpha¡\beta)_* * \vec{U}[j]^{\alpha-\beta} \right\} \right\} \tag{6.15}$$

The method of constants is substituted and the canonical form is compared to the transposed form.

$$\sum^{0\le l<L} \left\langle g[l](\vec{x}) = \sum^{\alpha\in E[l]} \{a[\alpha] * \vec{x}^{\alpha}\} \right\rangle \tag{6.16a}$$

$$\sum^{0\le l<L} \left\langle g[l](\vec{x}) = \sum^{\alpha\in E[l]} \left\{ a[\alpha] * \sum^{0\le k<K} \{w[l][k](\vec{x}) * H(\alpha;k)\} \right\} \right\rangle \tag{6.16b}$$

$$\sum^{0\le l<L} \left\langle g[l](\vec{x}) = \sum^{0\le k<K} \{w[l][k](\vec{x}) * c[k]\} \right\rangle \tag{6.16c}$$

The comparison results equations that determine the base polynomials.

$$\sum^{0\le l<L} \left\langle \sum^{\alpha\in E[l]} \left\langle \vec{x}^{\alpha} = \sum^{0\le k<K} \{w[l][k](\vec{x}) * H(\alpha;k)\} \right\rangle \right\rangle \tag{6.17}$$

A system of equations is determined for each polynomial.

$$\sum^{0\le l<L} \left\langle \sum^{\alpha\in E[l]} \left\langle \sum^{0\le k<K} \langle H(\alpha;k)\rangle \right\rangle * \sum^{0\le k<K} \langle w[l][k](\vec{x})\rangle = \sum^{\alpha\in E[l]} \langle \vec{x}^{\alpha}\rangle \right\rangle \tag{6.18}$$

The base matrices are defined.

$$\sum^{0\le l<L} \left\langle G[l] = \sum^{\alpha\in E[l]} \left\langle \sum^{0\le k<K} \langle H(\alpha;k)\rangle \right\rangle \right\rangle \tag{6.19}$$

A source matrix results from a base matrix by substituting one column by the source c which is equal for all polynomials.

$$\sum^{0\le l<L} \left\langle Q[l][m] = \sum^{\alpha\in E[l]} \left\langle \sum^{0\le k<K} \left\langle \begin{cases} \vec{x}^{\alpha} & \text{if } m = k \\ H(\alpha;k) & \text{otherwise} \end{cases} \right\rangle \right\rangle \right\rangle \tag{6.20}$$

A base polynomial is determined by Cramer's rule if the determinant of the corresponding base matrix is non-zero. Otherwise the base polynomial is undefined.

$$\sum^{0\le l<L} \left\langle \text{if}\,(\det(G[l]) \ne 0)\,\text{then} \left(w[l][k](\vec{x}) = \frac{\det(Q[l][m])}{\det(G[l])} \right) \right\rangle \tag{6.21}$$

The determinant of the largest absolute value of all determinants is defined by its index.

$$T = \text{iAbsMax} \sum^{0\le l<L} \langle\ \det(G[l])\ \rangle \tag{6.22}$$

A polynomial does not exist if the determinant of its base matrix equals zero. Therefore the polynomial of the largest determinant is selected as solution. Other criteria may apply.

$$\text{if}\,(\det(G[T]) \neq 0)\,\text{then}\left(f(\vec{x}) = \sum^{0\leq k<K} \{W[k](\vec{x}) * c[k]\}\,; \qquad \sum^{0\leq k<K} \langle W[k](\vec{x}) = w[T][k](\vec{x})\rangle \right) \tag{6.23}$$

6.1 Lagrange's Interpolation Formula

Lagrange's Interpolation formula results a transposed polynomial of a one-dimensional position and is determined by Dirichlet conditions. A function maps one value to one position. Therefore only one Dirichlet condition is allowed for any position. Thus positions and conditions are addressed by the the same index k.

$$\sum^{0\leq k<K} \left\langle \partial f[\mathbf{0}]\left(\vec{U}[k]\right) = y[k] \right\rangle ; \qquad K = \text{size}\left(\vec{U}[k]\right) \tag{6.24}$$

The transposed form of the polynomial is compared to its the canonical form.

$$f(\vec{x}) = \sum^{0\leq\alpha<A} \{a[\alpha] * \vec{x}^{\alpha}\} = \sum^{0\leq k<K} \left\{ w[k](\vec{x}) * \sum^{0\leq\alpha<A} \left\{ a[\alpha] * \vec{U}[k]^{\alpha} \right\} \right\} \tag{6.25}$$

The sums are interchanged.

$$f(\vec{x}) = \sum^{0\leq\alpha<A} \{a[\alpha] * \vec{x}^{\alpha}\} = \sum^{0\leq\alpha<A} \left\{ a[\alpha] * \sum^{0\leq k<K} \left\{ w[k](\vec{x}) * \vec{U}[k]^{\alpha} \right\} \right\} \tag{6.26}$$

The base polynomials are determined by a system of equations.

$$\sum^{0\leq\alpha<A} \left\langle \sum^{0\leq k<K} \left\langle \vec{U}[k]^{\alpha} \right\rangle \right\rangle * \sum^{0\leq k<K} \langle w[k](\vec{x})\rangle = \sum^{0\leq\alpha<A} \langle \vec{x}^{\alpha}\rangle \tag{6.27}$$

The base matrix is a Vandermonde matrix. Its determinant is non-zero if all positions are unique.

$$G = \sum^{0\leq\alpha<A} \left\langle \sum^{0\leq k<K} \left\langle \vec{U}[k]^{\alpha} \right\rangle \right\rangle \tag{6.28}$$

The determinant of a Vandermonde matrix equals the product of all possible differences.

$$\det(G) = \prod^{1\leq\alpha<A} \left\{ \prod^{0\leq\beta<\alpha} \left\{ \vec{U}[\alpha] - \vec{U}[\beta] \right\} \right\} \tag{6.29}$$

The determinants of the source matrices are defined accordingly.

$$\sum^{0\leq m<K} \left\langle \det(Q[m]) = \prod^{1\leq\alpha<A} \left\{ \prod^{0\leq\beta<\alpha} \left\{ \left(\begin{cases} \vec{x} - \vec{U}[\beta] & \text{if } \alpha = m \\ \vec{U}[\alpha] - \vec{x} & \text{if } \beta = m \\ \vec{U}[\alpha] - \vec{U}[\beta] & \text{otherwise} \end{cases} \right) \right\} \right\} \right\rangle \tag{6.30}$$

The base polynomials are determined by Cramer's rule if the determinant of the base matrix is non-zero.

$$\sum^{0\le m<K} \left\langle \text{if}\,(\det(G) \neq 0)\,\text{then} \left(w[m](\vec{x}) = \frac{\det(Q[m])}{\det(G)} = \frac{\prod\limits_{\alpha\neq m}^{0\le\alpha<A} \left\{\vec{U}[\alpha] - \vec{x}\right\}}{\prod\limits_{\alpha\neq m}^{0\le\alpha<A} \left\{\vec{U}[\alpha] - \vec{U}[m]\right\}} \right) \right\rangle \tag{6.31}$$

Lagrange's Interpolation formula is determined.

$$f(\vec{x}) = \sum^{0\le k<K} \{w[k](\vec{x}) * y[k]\} \tag{6.32}$$

6.2 Example of Derivatives of Base Polynomials

Three terms of a polynomial of a one-dimensional position are determined.

$$f(\vec{x}) = \sum^{\alpha\in A} \{a[\alpha] * \vec{x}^{\alpha}\}\,; \qquad A = (\langle 0\rangle\,; \langle 1\rangle\,; \langle 2\rangle) \tag{6.33}$$

The base polynomials are determined by a system of linear equations. The base matrix is a Vandermonde matrix.

$$\begin{bmatrix} \vec{u}[0]^0 & \vec{u}[1]^0 & \vec{u}[2]^0 \\ \vec{u}[0]^1 & \vec{u}[1]^1 & \vec{u}[2]^1 \\ \vec{u}[0]^2 & \vec{u}[1]^2 & \vec{u}[2]^2 \end{bmatrix} * \begin{bmatrix} w[0](\vec{u}) \\ w[1](\vec{u}) \\ w[2](\vec{u}) \end{bmatrix} = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \end{bmatrix} \tag{6.34}$$

The differentiation is applied to the system of linear equations.

$$\begin{bmatrix} \vec{u}[0]^0 & \vec{u}[1]^0 & \vec{u}[2]^0 \\ \vec{u}[0]^1 & \vec{u}[1]^1 & \vec{u}[2]^1 \\ \vec{u}[0]^2 & \vec{u}[1]^2 & \vec{u}[2]^2 \end{bmatrix} * \begin{bmatrix} \partial w[1][0](\vec{u}) \\ \partial w[1][1](\vec{u}) \\ \partial w[1][2](\vec{u}) \end{bmatrix} = \begin{bmatrix} 0 \\ x^0 \\ 2 * x^1 \end{bmatrix} \tag{6.35}$$

The zeroth source matrix is determined.

$$Q[0] = \begin{bmatrix} 0 & \vec{u}[1]^0 & \vec{u}[2]^0 \\ x^0 & \vec{u}[1]^1 & \vec{u}[2]^1 \\ 2 * x^1 & \vec{u}[1]^2 & \vec{u}[2]^2 \end{bmatrix} \tag{6.36}$$

The determinant is defined in terms of variants of a Vandermonde matrix, see section 2.15.

$$\det(Q[0]) = -x^0 * \det\left(\begin{bmatrix} \vec{u}[1]^0 & \vec{u}[2]^0 \\ \vec{u}[1]^2 & \vec{u}[2]^2 \end{bmatrix}\right) + 2 * \vec{x}^1 * \det\left(\begin{bmatrix} \vec{u}[1]^0 & \vec{u}[2]^0 \\ \vec{u}[1]^1 & \vec{u}[2]^1 \end{bmatrix}\right) \tag{6.37}$$

$$= \left(-\vec{x}^0 * \vec{x}[1]^1 * \vec{x}[2]^1 + 2 * \vec{x}^1\right) * \det\left(\begin{bmatrix} \vec{u}[1]^0 & \vec{u}[2]^0 \\ \vec{u}[1]^1 & \vec{u}[2]^1 \end{bmatrix}\right) \tag{6.38}$$

6.3 Examples of a One-Dimensional Poisson Equation

6.3.1 One Poisson and Two Dirichlet Conditions

Three terms of a polynomial of a one-dimensional position are determined.

$$f(\vec{x}) = a[0] * \vec{x}^0 + a[1] * \vec{x}^1 + a[2] * \vec{x}^2 \tag{6.39}$$

The polynomial is to be constrained by a Poisson condition at the origin and the Dirichlet condition at two other locations.

$$\partial f[2]\left(\vec{0}\right) = c = 2 * a[2] \tag{6.40}$$

$$\partial f[0]\left(\vec{U}[1]\right) = y[1] = a[0] * \vec{U}[1]^0 + a[1] * \vec{U}[1]^1 + a[2] * \vec{U}[1]^2 \tag{6.41}$$

$$\partial f[0]\left(\vec{U}[2]\right) = y[2] = a[0] * \vec{U}[2]^0 + a[1] * \vec{U}[2]^1 + a[2] * \vec{U}[2]^2 \tag{6.42}$$

The coefficients are determined by a system of linear equations.

$$G * a = \begin{bmatrix} 0 & 0 & 2 \\ \vec{U}[1]^0 & \vec{U}[1]^1 & \vec{U}[1]^2 \\ \vec{U}[2]^0 & \vec{U}[2]^1 & \vec{U}[2]^2 \end{bmatrix} * \begin{bmatrix} a[0] \\ a[1] \\ a[2] \end{bmatrix} = \begin{bmatrix} c \\ y[1] \\ y[2] \end{bmatrix} = b \tag{6.43}$$

The determinant of the base matrix is non-zero if the two locations are unique.

$$\det(G) = 2 * \left(\vec{U}[2] - \vec{U}[1]\right) \tag{6.44}$$

The polynomial coefficients are determined.

$$a[0] = \frac{1}{2} * c * \vec{U}[1] * \vec{U}[2] + \frac{y[1] * \vec{U}[2] - y[2] * \vec{U}[1]}{\vec{U}[2] - \vec{U}[1]} \tag{6.45}$$

$$a[1] = -\frac{1}{2} * c * \left(\vec{U}[1] + \vec{U}[2]\right) + \frac{y[2] - y[1]}{\vec{U}[2] - \vec{U}[1]} \tag{6.46}$$

$$a[2] = \frac{1}{2} * c \tag{6.47}$$

Coefficient $a[1]$ does not depend on the Poisson source in case of symmetry.

$$\vec{U} = \vec{U}[1] = -\vec{U}[2] \tag{6.48a}$$

$$a[0] = -\frac{1}{2} * c * \vec{U}^2 + \frac{y[1] + y[2]}{2} \tag{6.48b}$$

$$a[1] = -\frac{y[2] - y[1]}{2 * \vec{U}} \tag{6.48c}$$

$$a[2] = \frac{1}{2} * c \tag{6.48d}$$

The conditions are multiplied by weights and added.

$$\begin{aligned} & c * w[0](\vec{x}) + y[1] * w[1](\vec{x}) + y[2] * w[2](\vec{x}) \\ = \ & (2 * a[2]) * w[0](\vec{x}) \\ & + \left(a[0] * \vec{U}[1]^0 + a[1] * \vec{U}[1]^1 + a[2] * \vec{U}[1]^2\right) * w[1](\vec{x}) \\ & + \left(a[0] * \vec{U}[2]^0 + a[1] * \vec{U}[2]^1 + a[2] * \vec{U}[2]^2\right) * w[2](\vec{x}) \end{aligned} \tag{6.49}$$

$$= a[0] * \vec{x}^0 + a[1] * \vec{x}^1 + a[2] * \vec{x}^2 \tag{6.50}$$

The system of equations that determines the weights follows by comparison. The base matrix of weights is the transposed base matrix of coefficients.

$$\begin{bmatrix} 0 & \vec{U}[1]^0 & \vec{U}[2]^0 \\ 0 & \vec{U}[1]^1 & \vec{U}[2]^1 \\ 2 & \vec{U}[1]^2 & \vec{U}[2]^2 \end{bmatrix} * \begin{bmatrix} w[0](\vec{x}) \\ w[1](\vec{x}) \\ w[2](\vec{x}) \end{bmatrix} = \begin{bmatrix} \vec{x}^0 \\ \vec{x}^1 \\ \vec{x}^2 \end{bmatrix} \tag{6.51}$$

The weights or base polynomials are determined.

$$w[0](\vec{x}) = \frac{1}{2} * \left(\vec{U}[1] * \vec{U}[2] * \vec{x}^0 - \left(\vec{U}[1] + \vec{U}[2]\right) * \vec{x}^1 + \vec{x}^2\right) \tag{6.52}$$

$$w[1](\vec{x}) = \frac{\vec{U}[2] - \vec{x}^1}{\vec{U}[2] - \vec{U}[1]} \tag{6.53}$$

$$w[2](\vec{x}) = \frac{\vec{x}^1 - \vec{U}[1]}{\vec{U}[2] - \vec{U}[1]} \tag{6.54}$$

The transposed polynomial is determined.

$$f(\vec{x}) = c * w[0](\vec{x}) + y[1] * w[1](\vec{x}) + y[2] * w[2](\vec{x}) \tag{6.55}$$

6.3.2 Three Dirichlet Conditions

Three Dirichlet conditions are determined at unique locations.

$$\sum^{0 \le j < 3} \left\langle y[j] = a[0] + a[1] * \vec{U}[j] + a[2] * \vec{U}[j]^2 \right\rangle \tag{6.56}$$

The base polynomials are determined by a system of linear equations.

$$\begin{bmatrix} 1 & 1 & 1 \\ \vec{U}[0] & \vec{U}[1] & \vec{U}[2] \\ \vec{U}[0]^2 & \vec{U}[1]^2 & \vec{U}[2]^2 \end{bmatrix} * \begin{bmatrix} w[0](\vec{x}) \\ w[1](\vec{x}) \\ w[2](\vec{x}) \end{bmatrix} = \begin{bmatrix} \vec{x}^0 \\ \vec{x}^1 \\ \vec{x}^2 \end{bmatrix} \tag{6.57}$$

The differentiation is applied to the system of equations.

$$G * \partial w = \begin{bmatrix} 1 & 1 & 1 \\ \vec{U}[0] & \vec{U}[1] & \vec{U}[2] \\ \vec{U}[0]^2 & \vec{U}[1]^2 & \vec{U}[2]^2 \end{bmatrix} * \begin{bmatrix} \partial w[0][2](\vec{u}) \\ \partial w[1][2](\vec{u}) \\ \partial w[2][2](\vec{u}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = b \tag{6.58}$$

The base matrix is a Vandermonde matrix. The determinant equals the product of all possible differences.

$$\det(G) = \left(\vec{U}[1] - \vec{U}[0]\right) * \left(\vec{U}[2] - \vec{U}[0]\right) * \left(\vec{U}[2] - \vec{U}[1]\right) \tag{6.59}$$

The second derivative of the base polynomials are determined.

$$\partial w[0][2](\vec{u}) = \frac{2}{\left(\vec{U}[1] - \vec{U}[0]\right) * \left(\vec{U}[2] - \vec{U}[0]\right)} \tag{6.60}$$

$$\partial w[1][2](\vec{u}) = \frac{-2}{\left(\vec{U}[1] - \vec{U}[0]\right) * \left(\vec{U}[2] - \vec{U}[1]\right)} \tag{6.61}$$

$$\partial w[2][2](\vec{u}) = \frac{2}{(\vec{[U]})\, 2 - \vec{U}[0] * \left(\vec{U}[2] - \vec{U}[1]\right)} \tag{6.62}$$

The second derivative is constant.

$$\partial f[2](\vec{u}) = y[0] * \partial w[0][2](\vec{u}) + y[1] * \partial w[1][2](\vec{u}) + y[1] * \partial w[2][2](\vec{u}) = \text{const} \tag{6.63}$$

The solution is given at the origin.

$$\vec{U}[0] = \vec{0} \tag{6.64a}$$

$$\partial f[2](\vec{u}) = \frac{2 * y[0]}{\vec{U}[1] * \vec{U}[2]} - \frac{2 * y[1]}{\vec{U}[1] * \left(\vec{U}[2] - \vec{U}[1]\right)} + \frac{2 * y[2]}{\vec{U}[2] * \left(\vec{U}[2] - \vec{U}[1]\right)} \tag{6.64b}$$

An operator of Finite Differences results under symmetry.

$$\vec{V} = \vec{U}[1] = -\vec{U}[2] \tag{6.64c}$$

$$\partial f[2](\vec{u}) = -\frac{2 * y[0]}{\vec{V}^2} + \frac{y[1]}{\vec{V}^2} + \frac{y[2]}{\vec{V}^2} \tag{6.64d}$$

6.4 Examples of a Two-Dimensional Poisson Equation

6.4.1 Six Points

Six Dirichlet conditions are determined at unique locations of two dimensions.

$$\sum^{0 \le j < 6} \left\langle y[j] = \partial f[\mathbf{0}]\left(\vec{U}[j]\right)\right\rangle \tag{6.65}$$

A Poisson condition is determined.

$$\partial f[\langle 0;2\rangle](\vec{u}) + \partial f[\langle 2;0\rangle](\vec{u}) = 2 * a[\langle 0;2\rangle] + 2 * a[\langle 2;0\rangle] = c \tag{6.66}$$

The Poisson condition intersects the Dirichlet conditions.

$$2 * a[\langle 0;2\rangle] + 2 * a[\langle 2;0\rangle] = \sum^{0 \le j < 6} \left\{ w[j](\vec{x}) * \partial f[\mathbf{0}]\left(\vec{U}[j]\right)\right\} \tag{6.67}$$

The base polynomials are determined by a system of linear equations. The determinant of the base matrix is Zero if any two points are coincident or any four points map to line or all points map to a conic section.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \vec{U}[0]^{\langle 0;1\rangle} & \vec{U}[1]^{\langle 0;1\rangle} & \vec{U}[2]^{\langle 0;1\rangle} & \vec{U}[3]^{\langle 0;1\rangle} & \vec{U}[4]^{\langle 0;1\rangle} & \vec{U}[5]^{\langle 0;1\rangle} \\ \vec{U}[0]^{\langle 1;0\rangle} & \vec{U}[1]^{\langle 1;0\rangle} & \vec{U}[2]^{\langle 1;0\rangle} & \vec{U}[3]^{\langle 1;0\rangle} & \vec{U}[4]^{\langle 1;0\rangle} & \vec{U}[5]^{\langle 1;0\rangle} \\ \vec{U}[0]^{\langle 0;2\rangle} & \vec{U}[1]^{\langle 0;2\rangle} & \vec{U}[2]^{\langle 0;2\rangle} & \vec{U}[3]^{\langle 0;2\rangle} & \vec{U}[4]^{\langle 0;2\rangle} & \vec{U}[5]^{\langle 0;2\rangle} \\ \vec{U}[0]^{\langle 1;1\rangle} & \vec{U}[1]^{\langle 1;1\rangle} & \vec{U}[2]^{\langle 1;1\rangle} & \vec{U}[3]^{\langle 1;1\rangle} & \vec{U}[4]^{\langle 1;1\rangle} & \vec{U}[5]^{\langle 1;1\rangle} \\ \vec{U}[0]^{\langle 2;0\rangle} & \vec{U}[1]^{\langle 2;0\rangle} & \vec{U}[2]^{\langle 2;0\rangle} & \vec{U}[3]^{\langle 2;0\rangle} & \vec{U}[4]^{\langle 2;0\rangle} & \vec{U}[5]^{\langle 2;0\rangle} \end{bmatrix} * \begin{bmatrix} w[0](\vec{x}) \\ w[1](\vec{x}) \\ w[2](\vec{x}) \\ w[3](\vec{x}) \\ w[4](\vec{x}) \\ w[5](\vec{x}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 2 \end{bmatrix} \tag{6.68}$$

One point may be coincident with the origin and four other points map to the axes of the coordinate system.

$$\vec{U}[0] = \vec{\mathbf{0}}; \quad \vec{U}[1][0] = \vec{U}[2][1] = \vec{U}[3][0] = \vec{U}[4][1] = 0 \tag{6.69a}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \vec{U}[1]^{\langle 0;1\rangle} & 0 & \vec{U}[3]^{\langle 0;1\rangle} & 0 & \vec{U}[5]^{\langle 0;1\rangle} \\ 0 & 0 & \vec{U}[2]^{\langle 1;0\rangle} & 0 & \vec{U}[4]^{\langle 1;0\rangle} & \vec{U}[5]^{\langle 1;0\rangle} \\ 0 & \vec{U}[1]^{\langle 0;2\rangle} & 0 & \vec{U}[3]^{\langle 0;2\rangle} & 0 & \vec{U}[5]^{\langle 0;2\rangle} \\ 0 & 0 & 0 & 0 & 0 & \vec{U}[5]^{\langle 1;1\rangle} \\ 0 & 0 & \vec{U}[2]^{\langle 2;0\rangle} & 0 & \vec{U}[4]^{\langle 2;0\rangle} & \vec{U}[5]^{\langle 2;0\rangle} \end{bmatrix} * \begin{bmatrix} w[0](\vec{x}) \\ w[1](\vec{x}) \\ w[2](\vec{x}) \\ w[3](\vec{x}) \\ w[4](\vec{x}) \\ w[5](\vec{x}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 2 \end{bmatrix} \tag{6.69b}$$

In this case the base polynomial $w[5](\vec{u})$ equals Zero. Thus the source is neglected at this point and the polynomial is considered to be ill posed.

6.4.2 Seven Points

Seven terms of a polynomial are determined.

$$f(\vec{x}) = a[\langle 0;0\rangle] + a[\langle 0;1\rangle] * \vec{x}^{\langle 0;1\rangle} + a[\langle 1;1\rangle] * \vec{x}^{\langle 1;0\rangle} \tag{6.70}$$

$$+ a[\langle 0;2\rangle] * \vec{x}^{\langle 0;2\rangle} + a[\langle 1;1\rangle] * \vec{x}^{\langle 1;1\rangle} + a[\langle 2;0\rangle] * \vec{x}^{\langle 2;0\rangle} \tag{6.71}$$

$$g(\vec{x}) = f(\vec{x}) + a[\alpha] * \vec{x}^{\alpha} \tag{6.72}$$

Six points span a hexagon around the origin along the vertical axis.

i	1	2	3	4	5	6
$\vec{U}[i][0]$	d	0	$-d$	$-d$	0	d
$\vec{U}[i][1]$	d	$2*d$	d	$-d$	$-2*d$	$-d$

(6.73)

Seven base polynomials are determined by a system of linear equations.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & d & 2*d & d & -d & -2*d & -d \\ 0 & d & 0 & -d & -d & 0 & d \\ 2 & d^2 & 4*d^2 & d^2 & d^2 & 4*d^2 & d^2 \\ 0 & d^2 & 0 & -d^2 & d^2 & 0 & -d^2 \\ 2 & d^2 & 0 & d^2 & d^2 & 0 & d^2 \\ 0 & \vec{U}[1]^{\alpha} & \vec{U}[2]^{\alpha} & \vec{U}[3]^{\alpha} & \vec{U}[4]^{\alpha} & \vec{U}[5]^{\alpha} & \vec{U}[6]^{\alpha} \end{bmatrix} * \begin{bmatrix} w[\alpha][0] \\ w[\alpha][1] \\ w[\alpha][2] \\ w[\alpha][3] \\ w[\alpha][4] \\ w[\alpha][5] \\ w[\alpha][6] \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.74}$$

Four tuples of two dimensions and a horizontal check sum of Three exist.

$$\alpha \in (\quad \langle 0;3\rangle \,;\langle 1;2\rangle \,;\langle 2;1\rangle \,;\langle 3;0\rangle \quad) \tag{6.75}$$

The determinants of the four corresponding base matrices are given.

$$\det(G[\langle 0;3\rangle]) = -1536 * d^9; \qquad \det(G[\langle 1;2\rangle]) = 0 \tag{6.76}$$

$$\det(G[\langle 2;1\rangle]) = 512 * d^9; \qquad \det(G[\langle 3;0\rangle]) = 0 \tag{6.77}$$

Two polynomials of equal weights are available.

(6.78)

i	0	1	2	3	4	5	6
$w[\langle 0;3\rangle][i] = w[\langle 2;1\rangle][i]$	$-d^2/2$	1/4	0	1/4	1/4	0	1/4

Two weights equal Zero due to the two-fold symmetry. The polynomials are considered to be ill posed. The polynomial of the largest determinant is chosen as solution.

Chapter 7

Polynomial Approximation

A polynomial with an unknown array of tupels and coefficients is given.

$$g(\vec{x}) = \sum^{\gamma \in C} \{d[\gamma] * \vec{x}^{\gamma}\}; \qquad C \in \mathbb{T}\text{n}(n) \tag{7.1}$$

An array of unique constant positions is determined.

$$\sum^{0 \leq j < J} \left\langle \vec{U}[j] = \text{const} \right\rangle \tag{7.2}$$

At these locations K conditions on the polynomial are determined.

$$\sum^{0 \leq k < K} \left\langle \sum^{0 \leq j < J} \left\{ \sum^{\delta \in D} \left\{ b[k][j][\delta] * \partial g[\delta] \left(\vec{U}[j] \right) \right\} \right\} = c[k] \right\rangle; \qquad D = \mathbb{T}\text{nss}\left(n; \mathcal{G}\left(D\right)\right) \tag{7.3}$$

The polynomial is to be approximated with another polynomial of K terms.

$$f(\vec{x}) = \sum^{\alpha \in A} \{a[\alpha] * \vec{x}^{\alpha}\}; \qquad A \in \mathbb{T}\text{n}(n) \tag{7.4}$$

The accuracy of this approximation is determined by the transposed polynomial.

$$f(\vec{x}) = \sum^{0 \leq k < K} \{w[k] * c[k]\} \tag{7.5}$$

The polynomial is determined by the conditions such that the coefficients a are known.

$$\sum^{0 \leq k < K} \left\langle \sum^{0 \leq j < J} \left\{ \sum^{\beta \in B} \left\{ b[k][j][\beta] * \partial f[\beta] \left(\vec{U}[j] \right) \right\} \right\} = c[k] \right\rangle; \qquad B = \mathbb{T}\text{nss}\left(n; \mathcal{G}\left(A\right)\right) \tag{7.6}$$

A base polynomial or weight is determined by Cramer's rule according to (6.21).

$$\det(G) * f(\vec{x}) = \sum^{0 \leq k < K} \{\det(Q[k]) * c[k]\} \tag{7.7}$$

A source matrix is denoted explicitly.

$$Q[k] = \sum^{\substack{0\leq m<K\\ \alpha=A[m]}} \left\langle \sum^{0\leq l<K} \left\langle \begin{cases} \vec{x}^{\alpha} & \text{if } l=k \\ H(\alpha;l) & \text{otherwise} \end{cases} \right\rangle \right\rangle \tag{7.8a}$$

$$H(\alpha;k) = \sum^{0\leq j<J} \left\{ \sum_{\beta\leq\alpha}^{\beta\in B} \left\{ b[k][j][\beta] * (\alpha¡\beta)_{*} * \vec{U}[j]^{\alpha-\beta} \right\} \right\} \tag{7.8b}$$

The determinant of a source matrix is denoted by the sources as cofactors and submatrices.

$$Q[k][r] = \sum_{s\neq r}^{0\leq s<K} \left\langle \sum_{t\neq k}^{\substack{0\leq t<K\\ \alpha=A[t]}} \left\langle \; H(s;\alpha) \; \right\rangle \right\rangle \tag{7.8c}$$

$$\det(Q[k]) = \sum^{\substack{0\leq m<K\\ \alpha=A[i]}} \left\{ (-1)^{k+m} * \vec{x}^{\alpha} * \det(Q[k][m]) \right\} \tag{7.8d}$$

The derivatives of (7.3) are denoted explicitly according to (4.83).

$$\sum^{0\leq k<K} \left\langle \sum^{0\leq j<J} \left\{ \sum^{\delta\in D} \left\{ b[k][j][\delta] * \sum_{\gamma\geq\delta}^{\gamma\in C} \left\{ d[\gamma] * (\gamma¡\delta)_{*} * \vec{U}[j]^{\gamma-\delta} \right\} \right\} \right\} = c[k] \right\rangle \tag{7.9a}$$

The sums are rearranged such that all constants are elements of the inner sums.

$$\sum^{0\leq k<K} \left\langle \sum^{\gamma\in C} \left\{ d[\gamma] * \sum^{0\leq j<J} \left\{ \sum_{\delta\leq\gamma}^{\delta\in D} \left\{ b[k][j][\delta] * (\gamma¡\delta)_{*} * \vec{U}[j]^{\gamma-\delta} \right\} \right\} \right\} = c[k] \right\rangle \tag{7.9b}$$

The array of tupels B is defined as a subset of D. Coefficients of D may be Zero in order to fulfill this condition.

$$B \in D; \qquad B + E = D \tag{7.9c}$$

$$L(\gamma;k) = \sum^{0\leq j<J} \left\{ \sum_{\epsilon\leq\gamma}^{\epsilon\in E} \left\{ b[k][j][\epsilon] * (\gamma¡\epsilon)_{*} * \vec{U}[j]^{\gamma-\epsilon} \right\} \right\} \tag{7.9d}$$

Equations 7.9b is given in terms of methods.

$$\sum^{0\leq k<K} \left\langle \sum^{\gamma\in C} \left\{ d[\gamma] * (H(\gamma;k) + L(\gamma;k)) \right\} = c[k] \right\rangle \tag{7.9e}$$

Equation 7.7 is expanded by (7.8d) and (7.9e).

$$\begin{aligned} &\det(G) * f(\vec{x}) \\ = &\sum^{0\leq k<K} \left\{ \sum^{\substack{0\leq m<K\\ \alpha=A[m]}} \left\{ (-1)^{k+m} * \vec{x}^{\alpha} * \det(Q[k][m]) \right\} * \sum^{\gamma\in C} \left\{ d[\gamma] * (H(\gamma;k) + L(\gamma;k)) \right\} \right\} \end{aligned} \tag{7.10}$$

The equation is rearranged for the canonical form.

$$\det(G) * f(\vec{x}) = \sum_{\substack{0\le m<K \\ \alpha=A[m]}} \left\{ \vec{x}^{\alpha} * \sum^{\gamma\in C} \left\{ d[\gamma] * \sum^{0\le k<K} \left\{ (-1)^{k+m} * \det(Q[k][m]) * (H(\gamma;k) + L(\gamma;k)) \right\} \right\} \right\} \tag{7.11}$$

The innermost sum contains the determinant of the base matrix.

$$\sum^{0\le k<K} \left\{ (-1)^{k+m} * \det(Q[k][m]) * H(A[m]\,;k) \right\} = \det(G) \tag{7.12}$$

The innermost sum contains determinants of variants of the base matrix. These determinants equal Zero since two columns are identical.

$$\text{if}\,((\gamma \in A) * (\gamma \ne A[m]))\,\text{then}\left(\sum^{0\le k<K} \left\{ (-1)^{k+m} * \det(Q[k][m]) * H(\gamma;k) \right\} = 0 \right) \tag{7.13}$$

The remaining terms result a scalar which is determined by a method.

$$M(m;\gamma) = \sum^{0\le k<K} \left\{ (-1)^{k+m} * \frac{\det(Q[k][m])}{\det(G)} * L(\gamma;k) \right\} \tag{7.14}$$

The approximating polynomial is expressed in terms of the unknown coefficients.

$$f(\vec{x}) = \sum_{\substack{0\le m<K \\ \alpha=A[m]}} \left\{ \vec{x}^{\alpha} * \left(d[\alpha] + \sum_{\gamma>\alpha}^{\gamma\in C} \{ d[\gamma] * M(m;\gamma) \} \right) \right\} \tag{7.15}$$

The two polynomials are identical only if the array of coefficients is equal. Otherwise the two polynomials intersect only at the source points.

$$\text{if}\,(\text{size}\,(C) \ne 0)\,\text{then}\left(\sum_{\substack{0\le m<K \\ \alpha=A[m]}} \left\langle a[\alpha] = d[\alpha] + \sum_{\gamma>\alpha}^{\gamma\in C} \{ d[\gamma] * M(m;\gamma) \} \ne d[\alpha] \right\rangle \right) \tag{7.16}$$

The difference of the two polynomials is a constant if the arrays are not equal. Therefore the polynomial approximation is not accurate in this form.

$$\text{if}\,(\text{size}\,(C) \ne 0)\,\text{then}\,((f(\vec{x}) - g(\vec{x})) \in \mathbb{R}) \tag{7.17}$$

7.1 Example of Two Non-Intersecting Polynomials

Two differential conditions are determined on one dimension.

$$\partial f[0](-3) - \partial f[0](0) = -2 \tag{7.18}$$

$$\partial f[1](-1) = 1 \tag{7.19}$$

One polynomial of two terms fulfills these conditions.

$$f(\vec{x}) = -1 + x \tag{7.20}$$

At least one polynomial of three terms fulfills these conditions.

$$g(\vec{x}) = 2 + 3 * x + x^2 \tag{7.21}$$

The polynomials do not intersect and the first order polynomial is at no point an accurate approximation of the second order polynomial.

$$f(X) = -1 + X = 2 + 3 * X + X^2 = g(X)\,; \qquad X \notin \mathbb{R} \tag{7.22}$$

7.2 Local Approximation

A local approximation is defined here to require at least one Dirichlet-Condition e.

$$f\Big(\vec{U}[e]\Big) = g\Big(\vec{U}[e]\Big)\,; \qquad c[e] = \partial f[\mathbf{0}]\Big(\vec{U}[e]\Big) \tag{7.23}$$

The position is expressed in terms of the location of that Dirichlet-Condition.

$$f\Big(\vec{u} - \vec{U}[e]\Big) = \sum^{\substack{0 \leq m < K \\ \alpha = A[m]}} \left\{ \Big(\vec{u} - \overrightarrow{U}[e]\Big)^{\alpha} * \left(d[\alpha] + \sum_{\gamma > \alpha}^{\gamma \in C} \{d[\gamma] * M(m;\gamma)\}\right)\right\} \tag{7.24}$$

The binomial expansion is applied to position $\vec{u}$.

$$f\Big(\vec{u} - \vec{U}[e]\Big) = \sum^{\substack{0 \leq m < K \\ \alpha = A[m]}} \left\{ \sum_{\lambda \leq \alpha}^{\lambda \in B} \left\{ \binom{\alpha}{\lambda} * \vec{u}^{\lambda} * \vec{U}[e]^{\alpha - \lambda} \right\} * \left(d[\alpha] + \sum_{\gamma > \alpha}^{\gamma \in C} \{d[\gamma] * M(m;\gamma)\}\right)\right\} \tag{7.25}$$

The approximation depends on powers of the position of the Dirichlet-Condition. This dependency cancels if the origin and the position are coincident.

$$\vec{\mathbf{0}} = \vec{U}[e] \tag{7.26}$$

$$f(\vec{u}) = \sum^{\substack{0 \leq m < K \\ \alpha = A[m]}} \left\{ \vec{u}^{\alpha} * \left(d[\alpha] + \sum_{\gamma > \alpha}^{\gamma \in C} \{d[\gamma] * M(m;\gamma)\}\right)\right\} \tag{7.27}$$

The approximation is separated into two polynomials.

$$f(\vec{u}) = \sum^{\substack{0 \leq m < K \\ \alpha = A[m]}} \{\vec{u}^{\alpha} * d[\alpha]\} + \sum^{\substack{0 \leq m < K \\ \alpha = A[m]}} \left\{ \vec{u}^{\alpha} * \sum_{\gamma > \alpha}^{\gamma \in C} \{d[\gamma] * M(m;\gamma)\} \right\} = p(\vec{u}) + q(\vec{u}) \tag{7.28}$$

The approximation tends to the original polynomial if the value of polynomial $q(\vec{u})$ tends to Zero. The two polynomials are evaluated in relation to different origins.

$$\lim_{q(\vec{u}) \to 0} p(\vec{u}) = g(\vec{x}) \tag{7.29}$$

The order of a term of $q(u)$ equals the horizontal check sum of its tupel.

$$\mathcal{S}(d[\gamma] * M(m;\gamma)) = \mathcal{S}(\gamma) \tag{7.30}$$

The smallest horizontal check sum of the remainder polynomial is determined.

$$z = \min\left(\sum^{\gamma\in C} \langle \mathcal{S}(\gamma) \rangle\right) \tag{7.31}$$

The value of the remainder polynomial $q(\vec{u})$ depends mostly on terms of the smallest horizontal check sum z if the position $\vec{u}$ tends to Zero. Therefore it is required for sufficient accuracy that an array of tupels of a polynomial is complete such that the smallest horizontal check sum of the remainder is greater than the largest complete horizontal check sum of the approximating polynomial.

Chapter 8

Rational Functions

Two polynomials of the same position are determined.

$$p(\vec{x}) = \sum^{\alpha \in A} \{a[\alpha] * \vec{x}^{\alpha}\}; \qquad A \in \mathbb{T}\mathrm{n}(n) \tag{8.1}$$

$$q(\vec{x}) = \sum^{\beta \in B} \left\{b[\beta] * \vec{x}^{\beta}\right\}; \qquad B \in \mathbb{T}\mathrm{n}(n) \tag{8.2}$$

The second polynomial $q(\vec{x})$ has a finite number of real roots Z.

$$\sum^{0 \le i < \mathrm{size}(Z)} \left\langle q\left(\vec{Z}[i]\right) = 0\right\rangle \tag{8.3}$$

The quotient of the two polynomials is defined if the denominator does not equal Zero and thus the position does not equal a root of the second polynomial.

$$f(\vec{x}) = \frac{p(\vec{x})}{q(\vec{x})} = \frac{\sum^{\alpha \in A} \{a[\alpha] * \vec{x}^{\alpha}\}}{\sum^{\beta \in B} \left\{b[\beta] * \vec{x}^{\beta}\right\}}; \qquad \vec{x} \notin \vec{Z} \tag{8.4}$$

8.1 Derivatives

The derivatives of the polynomials are determined.

$$\sum^{\gamma \in C} \langle \partial p[\gamma](\vec{u}) = \partial\left(\gamma; p(\vec{x}); \vec{u}\right)\rangle; \qquad C = \mathbb{T}\mathrm{nss}\left(n; \mathcal{G}\left(A\right)\right) \tag{8.5}$$

$$\sum^{\delta \in D} \langle \partial q[\delta](\vec{u}) = \partial\left(\delta; q(\vec{x}); \vec{u}\right)\rangle; \qquad D = \mathbb{T}\mathrm{nss}\left(n; \mathcal{G}\left(B\right)\right) \tag{8.6}$$

The derivatives of a rational function are defined such that they map onto derivatives of polynomials, see section 8.5. The rational derivatives are only defined on domains that do not include roots of the denominator.

$$g(\vec{u};\vec{v}) = \frac{\sum^{\gamma\in C}\left\{\frac{\vec{v}^{\gamma}}{\gamma!_*} * \partial p[\gamma](\vec{u})\right\}}{\sum^{\delta\in D}\left\{\frac{\vec{v}^{\delta}}{\delta!_*} * \partial q[\delta](\vec{u})\right\}}; \qquad \vec{x} = \vec{u} + \vec{v}; \qquad \sum^{0\leq i<\mathrm{size}(\vec{x})} \left\langle\ 0 \notin \mathbb{R}\mathrm{AB}\left(\vec{u}[i]\,;\vec{x}[i]\right)\ \right\rangle \tag{8.7}$$

Derivatives of a rational function of higher orders are lengthy. Therefore only the derivatives of first and second order are defined. Derivatives of higher order may also be computed by repeated differentiation.

8.2 Differential Division

The derivatives of a rational function are determined by a repeated separation of the quotient according to section 2.3. The separation is applied only to quotients of which the numerator is a part of the original numerator $p(\vec{x})$. The separation is applied by terms of a smallest horizontal check sum.

A method is defined that gives the quotient of all derivatives of the numerator of a horizontal check sum of s and the derivative of the denominator of a horizontal check sum of Zero.

$$\mathrm{Ps}(s) = \frac{\sum_{\mathcal{S}(\gamma)=s}^{\gamma\in C}\left\{\frac{\vec{v}^{\gamma}}{\gamma!_*} * \partial p[\gamma](\vec{u})\right\}}{\sum_{\mathcal{S}(\delta)=0}^{\delta\in D}\left\{\frac{\vec{v}^{\delta}}{\delta!_*} * \partial q[\delta](\vec{u})\right\}} = \frac{\sum_{\mathcal{S}(\gamma)=s}^{\gamma\in C}\left\{\frac{\vec{v}^{\gamma}}{\gamma!_*} * \partial p[\gamma](\vec{u})\right\}}{\partial q[\mathbf{0}](\vec{u})} \tag{8.8}$$

A method is defined that gives the quotient of all derivatives of the denominator of a horizontal check sum of s and the derivative of the denominator of a horizontal check sum of Zero.

$$\mathrm{Qs}(s) = \frac{\sum_{\mathcal{S}(\delta)=s}^{\delta\in D}\left\{\frac{\vec{v}^{\delta}}{\delta!_*} * \partial q[\delta](\vec{u})\right\}}{\sum_{\mathcal{S}(\delta)=0}^{\delta\in D}\left\{\frac{\vec{v}^{\delta}}{\delta!_*} * \partial q[\delta](\vec{u})\right\}} = \frac{\sum_{\mathcal{S}(\delta)=s}^{\delta\in D}\left\{\frac{\vec{v}^{\delta}}{\delta!_*} * \partial q[\delta](\vec{u})\right\}}{\partial q[\mathbf{0}](\vec{u})} \tag{8.9}$$

A method is defined that gives the quotient of all derivatives of the numerator with a horizontal check sum of at least s and all derivatives of the denominator.

$$\mathrm{PS}(s) = \frac{\sum_{\mathcal{S}(\gamma)\geq s}^{\gamma\in C}\left\{\frac{\vec{v}^{\gamma}}{\gamma!_*} * \partial p[\gamma](\vec{u})\right\}}{\sum^{\delta\in D}\left\{\frac{\vec{v}^{\delta}}{\delta!_*} * \partial q[\delta](\vec{u})\right\}} = \frac{\sum_{\mathcal{S}(\gamma)\geq s}^{\gamma\in C}\left\{\frac{\vec{v}^{\gamma}}{\gamma!_*} * \partial p[\gamma](\vec{u})\right\}}{q(\vec{x})} \tag{8.10}$$

A method is defined that gives the quotient of all derivatives of the denominator with a horizontal check sum of at least s and all derivatives of the denominator.

$$\mathrm{QS}(s) = \frac{\sum\limits_{\mathcal{S}(\delta) \geq s}^{\delta \in D} \left\{ \frac{\vec{v}^{\delta}}{\delta!_*} * \partial q[\delta](\vec{u}) \right\}}{\sum^{\delta \in D} \left\{ \frac{\vec{v}^{\delta}}{\delta!_*} * \partial q[\delta](\vec{u}) \right\}} = \frac{\sum\limits_{\mathcal{S}(\delta) \geq s}^{\delta \in C} \left\{ \frac{\vec{v}^{\delta}}{\delta!_*} * \partial q[\delta](\vec{u}) \right\}}{q(\vec{x})} \tag{8.11}$$

The quotient of the derivatives of the polynomials is determined shortly.

$$g(\vec{u}; \vec{v}) = \mathrm{PS}(0) \tag{8.12}$$

Zeroth Separation

The quotient is separated by the derivatives of zero order.

$$\mathrm{PS}(0) = \mathrm{Ps}(0) + \mathrm{PS}(1) - \mathrm{Ps}(0) * \mathrm{QS}(1) \tag{8.13}$$

The terms of derivatives of zero order are defined.

$$\mathrm{Ds}[0] = \mathrm{Ps}(0) \tag{8.14}$$

The zeroth separation is redefined.

$$\mathrm{PS}(0) = \mathrm{Ds}[0] + \mathrm{PS}(1) - \mathrm{Ds}[0] * \mathrm{QS}(1) \tag{8.15}$$

First Separation

Two quotients are separated by the derivatives of first order.

$$\begin{aligned} \mathrm{PS}(0) = \mathrm{Ds}[0] &+ (\ \mathrm{Ps}(1) + \mathrm{PS}(2) - \mathrm{Ps}(1) * \mathrm{QS}(1)\ ) \\ &- \mathrm{Ds}[0] * (\ \mathrm{Qs}(1) + \mathrm{QS}(2) - \mathrm{Qs}(1) * \mathrm{QS}(1)\ ) \end{aligned} \tag{8.16}$$

The equation is rearranged by sums of terms of a horizontal check sum.

$$\mathrm{PS}(0) = \mathrm{Ds}[0] + (\ \mathrm{Ps}(1) - \mathrm{Ds}[0] * \mathrm{Qs}(1)\ ) * (\ 1 - \mathrm{QS}(1)\ ) + \mathrm{PS}(2) - \mathrm{Ds}[0] * \mathrm{QS}(1) \tag{8.17}$$

All quotients of first order derivatives and zero order derivatives are defined.

$$\mathrm{Ds}[1] = \mathrm{Ps}(1) - \mathrm{Ds}[0] * \mathrm{Qs}(1) \tag{8.18}$$

The first separation is redefined.

$$\mathrm{PS}(0) = \mathrm{Ds}[0] + \mathrm{Ds}[1] + \mathrm{PS}(2) - \mathrm{Ds}[1] * \mathrm{QS}(1) - \mathrm{Ds}[0] * \mathrm{QS}(1) \tag{8.19}$$

Second Separation

Three quotients are separated by the derivatives of second order.

$$\begin{aligned}\mathrm{PS}(0) = \mathrm{Ds}[0] + \mathrm{Ds}[1] + (\ \mathrm{Ps}(2) + \mathrm{PS}(3) - \mathrm{Ps}(2) * \mathrm{QS}(1)\ \ )\\ - \mathrm{Ds}[1] * (\ \mathrm{Qs}(1) + \mathrm{QS}(2) - \mathrm{Qs}(1) * \mathrm{QS}(1)\ \ )\\ - \mathrm{Ds}[0] * (\ \mathrm{Qs}(2) + \mathrm{QS}(3) - \mathrm{Qs}(2) * \mathrm{QS}(1)\ \ )\end{aligned} \tag{8.20}$$

All terms of second order derivatives and zero order derivatives are defined.

$$\mathrm{Ds}[2] = \mathrm{Ps}(2) - \mathrm{Ds}[1] * \mathrm{Qs}(1) - \mathrm{Ds}[0] * \mathrm{Qs}(2) \tag{8.21}$$

The second separation is redefined.

$$\mathrm{PS}(0) = \mathrm{Ds}[0] + \mathrm{Ds}[1] + \mathrm{Ds}[2] + \mathrm{PS}(3) - \mathrm{Ds}[2] * \mathrm{QS}(1) - \mathrm{Ds}[1] * \mathrm{QS}(2) - \mathrm{Ds}[0] * \mathrm{QS}(3) \tag{8.22}$$

Arbitrary Separation

All quotients of derivatives of order j and zero order derivatives are defined.

$$\mathrm{Ds}[j] = \mathrm{Ps}(j) - \sum^{0 \le i < j} \{\mathrm{Ds}[i] * \mathrm{Qs}(j-i)\} \tag{8.23}$$

The j-th separation is defined.

$$\mathrm{PS}(0) = \mathrm{PS}(J) + \sum^{0 \le j < J} \{\mathrm{Ds}[j] * (1 - \mathrm{QS}(J-j))\} \tag{8.24}$$

8.3 First Order Derivatives

The first separation is denoted by polynomial derivatives.

$$\mathrm{Ds}[1] = \frac{\sum^{\lambda \in L} \left\{ \frac{\vec{v}^{\lambda}}{\lambda!_*} * \partial p[\lambda](\vec{u}) \right\}}{\partial q[\mathbf{0}](\vec{u})} - \frac{\partial p[\mathbf{0}](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} * \frac{\sum^{\lambda \in L} \left\{ \frac{\vec{v}^{\lambda}}{\lambda!_*} * \partial q[\lambda](\vec{u}) \right\}}{\partial q[\mathbf{0}](\vec{u})}; \qquad L = \mathbb{T}\mathrm{ns}\,(n;1) \tag{8.25}$$

The terms are grouped by Taylor coefficients.

$$\mathrm{Ds}[1] = \sum^{\lambda \in L} \left\{ \frac{\vec{v}^{\lambda}}{\lambda!_*} * \left(\frac{\partial p[\lambda](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} - \frac{\partial p[\mathbf{0}](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} * \frac{\partial q[\lambda](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} \right) \right\} \tag{8.26}$$

The first order derivative σ of a rational function is determined.

$$\partial f[\sigma](\vec{u}) = \frac{\partial p[\sigma](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} - \frac{\partial p[\mathbf{0}](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} * \frac{\partial q[\sigma](\vec{u})}{\partial q[\mathbf{0}](\vec{u})}; \qquad \mathcal{S}(\sigma) = 1 \tag{8.27}$$

The quotient rule is determined.

$$\partial f[\sigma](\vec{u}) = \frac{\partial p[\sigma](\vec{u}) * \partial q[\mathbf{0}](\vec{u}) - \partial p[\mathbf{0}](\vec{u}) * \partial q[\sigma](\vec{u})}{\partial q[\mathbf{0}](\vec{u}) * \partial q[\mathbf{0}](\vec{u})}; \qquad \mathcal{S}(\sigma) = 1 \tag{8.28}$$

The first separation is denoted by rational derivatives.

$$\mathrm{Ds}[1] = \sum^{\lambda\in L} \left\{ \frac{\vec{v}^{\lambda}}{\lambda!_*} * \partial f[\lambda](\vec{u}) \right\}; \qquad L = \mathbb{T}\mathrm{ns}\,(n;1) \tag{8.29}$$

8.4 Second Order Derivatives

The second separation is denoted by polynomial derivatives.

$$\begin{aligned}
\mathrm{Ds}[2] = {} & \frac{\sum^{\mu\in M}\left\{\frac{\vec{v}^{\mu}}{\mu!_*} * \partial p[\mu](\vec{u})\right\}}{\partial q[\mathbf{0}](\vec{u})} \\
& - \frac{\sum^{\lambda\in L}\left\{\frac{\vec{v}^{\lambda}}{\lambda!_*} * \partial p[\lambda](\vec{u})\right\}}{\partial q[\mathbf{0}](\vec{u})} * \frac{\sum^{\kappa\in L}\left\{\frac{\vec{v}^{\kappa}}{\kappa!_*} * \partial q[\kappa](\vec{u})\right\}}{\partial q[\mathbf{0}](\vec{u})} \\
& + \frac{\partial p[\mathbf{0}](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} * \frac{\sum^{\lambda\in L}\left\{\frac{\vec{v}^{\lambda}}{\lambda!_*} * \partial q[\lambda](\vec{u})\right\}}{\partial q[\mathbf{0}](\vec{u})} * \frac{\sum^{\kappa\in L}\left\{\frac{\vec{v}^{\kappa}}{\kappa!_*} * \partial q[\kappa](\vec{u})\right\}}{\partial q[\mathbf{0}](\vec{u})} \\
& - \frac{\partial p[\mathbf{0}](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} * \frac{\sum^{\mu\in M}\left\{\frac{\vec{v}^{\mu}}{\mu!_*} * \partial q[\mu](\vec{u})\right\}}{\partial q[\mathbf{0}](\vec{u})}; \qquad \begin{array}{l} L = \mathbb{T}\mathrm{ns}\,(n;1) \\ M = \mathbb{T}\mathrm{ns}\,(n;2) \end{array}
\end{aligned} \tag{8.30}$$

The terms are grouped by Taylor coefficients.

$$\begin{aligned}
\mathrm{Ds}[2] = {} & \sum^{\mu\in M}\left\{\frac{\vec{v}^{\mu}}{\mu!_*} * \frac{\partial p[\mu](\vec{u})}{\partial q[\mathbf{0}](\vec{u})}\right\} \\
& - \sum^{\lambda\in L}\left\{\sum^{\kappa\in L}\left\{\frac{\vec{v}^{\lambda}}{\lambda!_*} * \frac{\vec{v}^{\kappa}}{\kappa!_*} * \frac{\partial p[\lambda](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} * \frac{\partial q[\kappa](\vec{u})}{\partial q[\mathbf{0}](\vec{u})}\right\}\right\} \\
& + \frac{\partial p[\mathbf{0}](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} * \sum^{\lambda\in L}\left\{\sum^{\kappa\in L}\left\{\frac{\vec{v}^{\lambda}}{\lambda!_*} * \frac{\vec{v}^{\kappa}}{\kappa!_*} * \frac{\partial q[\lambda](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} * \frac{\partial q[\kappa](\vec{u})}{\partial q[\mathbf{0}](\vec{u})}\right\}\right\} \\
& - \frac{\partial p[\mathbf{0}](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} * \sum^{\mu\in M}\left\{\frac{\vec{v}^{\mu}}{\mu!_*} * \frac{\partial q[\mu](\vec{u})}{\partial q[\mathbf{0}](\vec{u})}\right\}
\end{aligned} \tag{8.31}$$

A derivative of order σ is to be determined. Therefore index κ is expressed by the constant σ and the index λ.

$$\lambda + \kappa = \sigma; \qquad \kappa = \sigma - \lambda \tag{8.32}$$

The product of powers of distance are transformed into a product of a Taylor coefficient.

$$\frac{\vec{v}^{\lambda}}{\lambda!_*} * \frac{\vec{v}^{\kappa}}{\kappa!_*} = \frac{\vec{v}^{\sigma}}{\lambda!_* * (\sigma-\lambda)!_*} = \frac{\vec{v}^{\sigma}}{\sigma!_*} * \frac{\sigma!_*}{\lambda!_* * (\sigma-\lambda)!_*} \tag{8.33}$$

Set M contains only one element of tuple σ. A second derivative of a rational function is defined.

$$\begin{aligned}\partial f[\sigma](\vec{u}) = {} & \frac{\partial p[\sigma](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} \\ & - \sum_{\lambda \le \sigma}^{\lambda \in L} \left\{ \frac{\sigma!_*}{(\sigma-\lambda)!_* * \lambda!_*} * \frac{\partial p[\lambda](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} * \frac{\partial q[\sigma-\lambda](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} \right\} \\ & + \sum_{\lambda \le \sigma}^{\lambda \in L} \left\{ \frac{\sigma!_*}{(\sigma-\lambda)!_* * \lambda!_*} * \frac{\partial p[\mathbf{0}](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} * \frac{\partial q[\lambda](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} * \frac{\partial q[\sigma-\lambda](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} \right\} \\ & - \frac{\partial p[\mathbf{0}](\vec{u})}{\partial q[\mathbf{0}](\vec{u})} * \frac{\partial q[\sigma](\vec{u})}{\partial q[\mathbf{0}](\vec{u})}; \qquad \mathcal{S}(\sigma) = 2\end{aligned} \tag{8.34}$$

The second separation is denoted by rational derivatives.

$$\mathrm{Ds}[2] = \sum^{\lambda \in L} \left\{ \frac{\vec{v}^{\lambda}}{\lambda!_*} * \partial f[\lambda](\vec{u}) \right\}; \qquad L = \mathbb{T}\mathrm{ns}\,(n; 2) \tag{8.35}$$

8.5 Example

A rational function is determined.

$$f(x) = \frac{1}{1 + \vec{x}^2} \tag{8.36}$$

The polynomial division by terms of smallest orders is determined.

$$f(x) = 1 - \vec{x}^2 + \vec{x}^4 - \vec{x}^6 + g(x) \tag{8.37}$$

The only first derivative is defined by the quotient rule (8.28).

$$\partial f[0](\vec{u}) = -\frac{2 * \vec{u}}{1 + 2 * \vec{u}^2 + \vec{u}^4} \tag{8.38}$$

The polynomial division is applied to the first derivative. The result equals the derivative of the polynomial obtained by separating the rational function.

$$\partial f[0](\vec{u}) = -2 * \vec{u} + 4 * \vec{u}^3 - 6 * \vec{u}^5 + 8 * \vec{u}^7 + h(\vec{u}) \tag{8.39}$$

Chapter 9

Exponential Function

The exponential function is defined as the power of the universal constant $\mathtt{e}$ or Euler number.

$$\exp(\vec{x}) = \mathtt{e}^{\vec{x}}; \qquad \text{size}\left(\vec{x}\right) = 1 \tag{9.1}$$

Logarithm and exponential funcition are inverse.

$$\log[\mathtt{e}]\left(\mathtt{e}^{\vec{x}}\right) = \vec{x} \tag{9.2}$$

Any other power is determined by the exponential function, see [3, p67] or others for details.

$$a^{\vec{x}} = \mathtt{e}^{\log[\mathtt{e}](a) * \vec{x}} \tag{9.3}$$

The power of $\mathtt{e}$ determines one exact base point.

$$\exp\left(\vec{\mathbf{0}}\right) = \mathtt{e}^{\vec{\mathbf{0}}} = 1 \tag{9.4}$$

A number of continuous exponential conditions determine approximations to the power of $\mathtt{e}$.

$$\partial\left(\beta + \mathbf{1}; \mathtt{e}^{\vec{x}}; \vec{u}\right) - \partial\left(\beta; \mathtt{e}^{\vec{x}}; \vec{u}\right) = 0 \tag{9.5}$$

9.1 Single First Degree Extrapolation

The terms of an approximation polynomial and its first two derivatives are determined at the origin.

$$f(\vec{x}) = a[0] * \vec{x}^0 + a[1] * \vec{x}^1; \qquad \partial f[0]\left(\vec{\mathbf{0}}\right) = a[0]; \qquad \partial f[1]\left(\vec{\mathbf{0}}\right) = a[1] \tag{9.6}$$

The approximation of the exponential function is determined by two conditions.

$$\partial f[0]\left(\vec{\mathbf{0}}\right) = 1; \qquad \partial f[1]\left(\vec{\mathbf{0}}\right) - \partial f[0]\left(\vec{\mathbf{0}}\right) = 0 \tag{9.7}$$

The approximation is determined by the transposed polynomial.

$$y = f(\vec{x}) = w[0] * \partial f[0]\left(\vec{\mathbf{0}}\right) + w[1] * \left(\partial f[1]\left(\vec{\mathbf{0}}\right) - \partial f[0]\left(\vec{\mathbf{0}}\right)\right) = w[0] \tag{9.8}$$

The base polynomials or weights are determined by a system of linear equations.

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} w[0] \\ w[1] \end{bmatrix} = \begin{bmatrix} 1 \\ \vec{x} \end{bmatrix}; \qquad w[0] = 1 + \vec{x} \tag{9.9}$$

The approximation is only strictly stable within a radius of half a unit according to (2.48a).

$$\text{if}\left(\text{abs}(\vec{x}) < \frac{1}{2}\right) \text{then}\,(w[0] \text{ is strictly stable}) \tag{9.10}$$

A law of exponents applies for positions less than half a unit since terms of small orders dominate terms of higher orders. The radius is discussed below (9.15).

$$w[0]^k = \left(1 + \vec{h}\right)^k \approx \left(\mathsf{e}^{\vec{h}}\right)^k = \mathsf{e}^{k*\vec{h}} = \mathsf{e}^{\vec{x}}; \quad \text{abs}\left(k * \vec{h}\right) < \frac{1}{2}; \quad \text{abs}\left(\vec{h}^m\right) > \text{abs}\left(\vec{h}^{m+1}\right) \tag{9.11}$$

The square is determined.

$$\left(1 + \vec{h}\right)^2 = 1 + 2 * \vec{h} + \vec{h}^2 \tag{9.12a}$$

The series is checked against strict stability by terms.

$$\text{abs}\left(2 * \vec{h}\right) > \text{abs}\left(\vec{h}^2\right); \qquad \text{abs}\left(\vec{h}\right) < 1 \tag{9.12b}$$

$$\text{abs}(1) > \text{abs}\left(2 * \vec{h}\right) + \text{abs}\left(\vec{h}^2\right); \qquad \text{abs}\left(\vec{h}\right) < -1 + \sqrt{\frac{3}{2}} \approx 0.224 \tag{9.12c}$$

The intersection of strictly stable domains is determined.

$$\text{if}\left(\text{abs}\left(\vec{h}\right) < -1 + \sqrt{\frac{3}{2}}\right) \text{then}\left(\left(1 + \vec{h}\right)^2 \text{ is stable}\right) \tag{9.12d}$$

The radius of the square is less than the radius of the regular approximation.

$$k * h = 2 * \left(-1 + \sqrt{\frac{3}{2}}\right) \approx 0.448 < \frac{1}{2} \tag{9.12e}$$

The radius of self-accuracy of this approximation is determined by an inequality of the zeroth term.

$$1 > 2 * \left(\left(1 + \text{abs}\left(\vec{h}\right)\right)^j - 1\right) \tag{9.13}$$

A number of samples are computed with GMP [4] in C and a precision of 4096 bits.

j	2	5	10	100	1000	10000
$\vec{h} \lessapprox H$	0.224	0.0844	0.0413	0.00406	0.000405	0.0000405
$j * H$	0.448	0.422	0.413	0.406	0.405	0.405

(9.14)

It is assumed that the radius of self-accuracy tends to approximately 0.405. The power of the approximation converges slowly outside that radius. The radius of (9.11) is redefined.

$$w[0]^k = \left(1 + \vec{h}\right)^k \approx \left(\mathsf{e}^{\vec{h}}\right)^k = \mathsf{e}^{k*\vec{h}} = \mathsf{e}^{\vec{x}}; \quad \text{abs}\left(k \vec{*} h\right) \lessapprox 0.405; \quad \text{abs}\left(\vec{h}^m\right) > \text{abs}\left(\vec{h}^{m+1}\right) \tag{9.15}$$

The inverse is determined by the logarithm. An approximation of the logarithm is evaluated within its stable domain of position 1/3 as an example. The approximation of this inverse results a polynomial of second degree and does not determine the accuracy.

$$\log[\mathsf{e}]\left[\frac{1}{3}\right](\vec{z}) = \log[\mathsf{e}]\left(\frac{1}{3}\right) + \sum^{1\leq j<3} \left\{(-1)^{j-1} * \frac{\left(\vec{z} - \frac{1}{3}\right)^j}{j * \frac{1}{3^j}}\right\}; \qquad \frac{1}{6} < \vec{z} < \frac{1}{2} \tag{9.16a}$$

$$= \log[\mathsf{e}]\left(\frac{1}{3}\right) - \frac{3}{2} + 6 * \vec{z} - \frac{9}{2} * \vec{z}^2 \tag{9.16b}$$

$$\log[\mathsf{e}]\left[\frac{1}{3}\right](w[0]) = \log[\mathsf{e}]\left[\frac{1}{3}\right](1 + \vec{x}) = \log[\mathsf{e}]\left(\frac{1}{3}\right) - 3 * \vec{x} - \frac{9}{2} * \vec{x}^2 \tag{9.16c}$$

9.2 Single Extrapolation

The terms of a polynomial of a one-dimensional position are determined.

$$f(\vec{x}) = \sum^{0\leq i<n} \left\{a[i] * \vec{x}^i\right\} \tag{9.17}$$

The polynomial is determined by a Dirichlet and a number of exponential conditions.

$$\partial f[0]\left(\vec{\mathbf{0}}\right) = 1; \qquad \sum^{0\leq i<n-1} \left\langle \partial f[i]\left(\vec{\mathbf{0}}\right) - \partial f[i+1]\left(\vec{\mathbf{0}}\right) = 0\right\rangle \tag{9.18}$$

The terms of the transposed polynomial are determined.

$$f(\vec{x}) = w[0](\vec{x}) * \partial f[0]\left(\vec{\mathbf{0}}\right) + \sum^{1\leq i<n} \left\{w[i](\vec{x}) * \left(\partial f[i]\left(\vec{\mathbf{0}}\right) - \partial f[i+1]\left(\vec{\mathbf{0}}\right)\right)\right\} \tag{9.19}$$

The weights or base polynomials are determined by a system of linear equations.

$$G * w = \begin{bmatrix} 1! & -1! & 0 & 0 & 0 & \dots \\ 0 & 1! & -1! & 0 & 0 & \dots \\ 0 & 0 & 2! & -2! & 0 & \dots \\ 0 & 0 & 0 & 3! & -3! & \dots \\ 0 & 0 & 0 & 0 & 4! & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} * \begin{bmatrix} w[0](\vec{x}) \\ w[1](\vec{x}) \\ w[2](\vec{x}) \\ w[3](\vec{x}) \\ w[4](\vec{x}) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ \vec{x} \\ \vec{x}^2 \\ \vec{x}^3 \\ \vec{x}^4 \\ \vdots \end{bmatrix} \tag{9.20}$$

The determinant equals a super faculty and contains at least all primes up to n.

$$\det(G) = \prod^{0\leq i<n} \{i!\} \tag{9.21}$$

The transposed polynomial is determined by the only non-zero condition.

$$f(\vec{x}) = w[0](\vec{x}) = \sum^{0\leq i<n} \left\{ \frac{\vec{x}^i}{i!} \right\} = \left(1 + \frac{\vec{x}}{1} * \left(1 + \frac{\vec{x}}{2} * \left(1 + \frac{\vec{x}}{3} * (...)\right)\right)\right) \tag{9.22}$$

Strict stability is determined by an inequality.

$$\text{abs}\left(\frac{\vec{x}^i}{i!}\right) > 2 * \text{abs}\left(\frac{\vec{x}^{i+1}}{(i+1)!}\right); \qquad \text{abs}(\vec{x}) < \frac{i+1}{2} \tag{9.23}$$

Strict stability is determined by the zeroth term.

$$\text{if}\left(\text{abs}(\vec{x}) < \frac{1}{2}\right) \text{then}\left(f(\vec{x}) \text{ is strictly stable}\right) \tag{9.24}$$

A law of exponents applies for positions less than unity since terms of small orders dominate terms of higher orders. The radius of self-accuracy is determined by the condition on the zeroth term (9.15).

$$w[0]^k \approx \left(\mathsf{e}^{\vec{h}}\right)^k = \mathsf{e}^{k*\vec{h}} = \mathsf{e}^{\vec{x}}; \qquad \text{abs}\left(k * \vec{h}\right) \lessapprox 0.405; \qquad \text{abs}\left(\vec{h}^m\right) > \text{abs}\left(\vec{h}^{m+1}\right) \tag{9.25}$$

9.3 Double First Degree Extrapolation

The terms of an approximation polynomial are determined.

$$f(\vec{x}) = a[0] * \vec{x}^0 + a[1] * \vec{x}^1 + a[2] * \vec{x}^2 \tag{9.26}$$

The initial two derivatives are polynomials of $\vec{u}$.

$$\partial f[0](\vec{u}) = a[0] * \vec{u}^0 + a[1] * \vec{u}^1 + a[2] * \vec{u}^2 \tag{9.27}$$

$$\partial f[1](\vec{u}) = a[1] + 2 * a[2] * \vec{u} \tag{9.28}$$

The polynomial is determined by a Dirichlet condition at the origin and and a exponential condition at the origin and at the extrapolation point $\vec{x}$.

$$\partial f[0](\vec{u}) = 1; \qquad \partial f[1]\left(\vec{\mathbf{0}}\right) - \partial f[0]\left(\vec{\mathbf{0}}\right) = 0; \qquad \partial f[1](\vec{x}) - \partial f[0](\vec{x}) = 0 \tag{9.29}$$

The transposed polynomial determines the value at the extrapolation point by the only non-zero condition. The weights or base polynomials are determined by a system of linear equations.

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 - \vec{x} \\ 0 & 0 & 2 * \vec{x} - \vec{x}^2 \end{bmatrix} * \begin{bmatrix} w[0](\vec{x}) \\ w[1](\vec{x}) \\ w[2](\vec{x}) \end{bmatrix} = \begin{bmatrix} 1 \\ \vec{x} \\ \vec{x}^2 \end{bmatrix}; \qquad w[0](\vec{x}) = \frac{2 + \vec{x}}{2 - \vec{x}} \tag{9.30}$$

Numerator and denominator are strictly stable within a radius of unity. A division of two stable series is assumed to be strictly stable.

$$\text{if}\left(\text{abs}(x) < 1\right) \text{then}\left(w[0](\vec{x}) \text{ is strictly stable}\right) \tag{9.31}$$

The repeated separation by smallest orders results the initial three terms of the single approximation (9.22). The radius of strict stability of single approximation is only half a unit (9.24).

$$\frac{2+\vec{x}}{2-\vec{x}} = 1 + \vec{x} + \frac{\vec{x}^2}{2} + \frac{\vec{x}^3}{4} + \frac{\vec{x}^4}{4} * \frac{1}{2-\vec{x}} \tag{9.32}$$

A law of exponents applies for positions less than a unit since terms of small orders dominate terms of higher orders.

$$w[0]^k = \left(\frac{2+\vec{h}}{2-\vec{h}}\right)^k \approx \left(\mathsf{e}^{\vec{h}}\right)^k = \mathsf{e}^{k*\vec{h}} = \mathsf{e}^{\vec{x}}; \quad \text{abs}\left(k*\vec{h}\right) < 1; \quad \text{abs}\left(\vec{h}^m\right) > \text{abs}\left(\vec{h}^{m+1}\right) \tag{9.33}$$

The square of a factor is determined.

$$(2 \pm \vec{x})^2 = 4 \pm 4 * \vec{x} + \vec{x}^2 \tag{9.34a}$$

The series is checked against strict stability by terms.

$$\text{abs}(4*\vec{x}) > 2*\text{abs}\left(\vec{x}^2\right); \qquad \text{abs}(\vec{x}) < 2 \tag{9.34b}$$

$$4 > 2*\left(\text{abs}(\pm 4*\vec{x}) + \text{abs}\left(\vec{x}^2\right)\right); \qquad \text{abs}(\vec{x}) < -2+\sqrt{6} \tag{9.34c}$$

The intersection of strictly stable domains is determined.

$$\text{if}\left(\text{abs}\left(\vec{h}\right) < -2+\sqrt{6}\right) \text{then} \left(\left(1+\vec{h}\right)^2 \text{ is stable}\right) \tag{9.34d}$$

The radius of the square is less than the radius of the regular approximation.

$$k*h = 2*\left(-2+\sqrt{6}\right) \approx 2*0.449 = 0.898 < 1 \tag{9.34e}$$

The radius of self-accuracy of this approximation is determined by an inequality of the zeroth term.

$$2^j > 2*\left(\left(2+\text{abs}\left(\pm\vec{h}\right)\right)^j - 2^j\right) \tag{9.35}$$

A number of samples are computed with GMP [4] in C and a precision of 65536 bits.

j	2	5	10	100	1000	10000
$\vec{h} \lessapprox H$	0.449	0.168	0.0827	0.00812	0.000811	0.0000810
$j*H$	0.898	0.840	0.827	0.812	0.811	0.810

(9.36)

It is assumed that the radius of self-accuracy tends to approximately 0.810. The power of the approximation converges slowly outside that radius. The radius of (9.33) is redefined.

$$w[0]^k = \left(1+\vec{h}\right)^k \approx \left(\mathsf{e}^{\vec{h}}\right)^k = \mathsf{e}^{k*\vec{h}} = \mathsf{e}^{\vec{x}}; \quad \text{abs}\left(k \vec{*} h\right) \lessapprox 0.810; \quad \text{abs}\left(\vec{h}^m\right) > \text{abs}\left(\vec{h}^{m+1}\right) \tag{9.37}$$

9.4 Double Extrapolation

The terms of a polynomial of a one-dimensional position are determined.

$$f(\vec{x}) = \sum^{0\le i\le 2*n} \left\{a[i]*\vec{x}^i\right\} \tag{9.38}$$

The polynomial is determined by a Dirichlet and a number of exponential conditions at the origin and the extrapolation point $\vec{x}$.

$$\partial f[0]\left(\vec{\mathbf{0}}\right) = 1 \tag{9.39a}$$

$$\sum^{0\le i<n} \left\langle \partial f[i]\left(\vec{\mathbf{0}}\right) - \partial f[i+1]\left(\vec{\mathbf{0}}\right) = 0\right\rangle; \qquad \sum^{0\le i<n} \langle \partial f[i](\vec{x}) - \partial f[i+1](\vec{x}) = 0\rangle \tag{9.39b}$$

The transposed polynomial is determined by a system of linear equations of base polynomials $w[i](\vec{x})$.

$$\left[\begin{array}{cccccccc|c}
1 & -1 & -1 & 0 & 0 & 0 & 0 & \dots & 1 \\
0 & 1 & 1-\vec{x} & -1 & -1 & 0 & 0 & \dots & \vec{x} \\
0 & 0 & 2*\vec{x}-\vec{x}^2 & 2 & 2-2*\vec{x} & -2 & -2 & \dots & \vec{x}^2 \\
0 & 0 & 3*\vec{x}^2-\vec{x}^3 & 0 & 6*\vec{x}-3*\vec{x}^2 & 6 & 6-6*\vec{x} & \dots & \vec{x}^3 \\
0 & 0 & 4*\vec{x}^3-\vec{x}^4 & 0 & 12*\vec{x}^2-4*\vec{x}^3 & 0 & 24*\vec{x}-12*\vec{x}^2 & \dots & \vec{x}^4 \\
0 & 0 & 5*\vec{x}^4-\vec{x}^5 & 0 & 20*\vec{x}^3-5*\vec{x}^4 & 0 & 60*\vec{x}^2-20*\vec{x}^3 & \dots & \vec{x}^5 \\
0 & 0 & 6*\vec{x}^5-\vec{x}^6 & 0 & 30*\vec{x}^4-6*\vec{x}^5 & 0 & 120*\vec{x}^3-30*\vec{x}^4 & \dots & \vec{x}^6 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots
\end{array}\right] \tag{9.40}$$

The transposed polynomial is determined by the zeroth base polynomial.

$$f(\vec{x}) = w[0](\vec{x}) = \frac{\sum^{0\le i\le n}\left\{\frac{(2*n-i)!}{(n-i)!*i!}*\vec{x}^i\right\}}{\sum^{0\le i\le n}\left\{(-1)^i*\frac{(2*n-i)!}{(n-i)!*i!}*\vec{x}^i\right\}} \tag{9.41}$$

The polynomial division by the smallest orders results the initial terms of a single extrapolation (9.22) and a remainder which may be separated such that all polynomial and rational terms are of at least $(2*n)$-th order.

$$f(\vec{x}) = \sum^{0\le i<2*n}\left\{\frac{\vec{x}^i}{i!}\right\} + \mathcal{O}\left(\vec{x}^{2*n}\right) \tag{9.42}$$

Strict stability is determined by an inequality.

$$\frac{(2*n-i)!}{(n-i)!*i!}*\mathrm{abs}\left(\vec{x}^i\right) > 2*\frac{(2*n-(i+1))!}{(n-(i+1))!*(i+1)!}*\mathrm{abs}\left(\vec{x}^{i+1}\right) \tag{9.43}$$

$$\frac{1}{2}*\frac{(2*n-i)!}{(2*n-(i+1))!}*\frac{(n-(i+1))!}{(n-i)!}*\frac{(i+1)!}{i!} > \mathrm{abs}(x) \tag{9.44}$$

$$\frac{1}{2}*(2*n-i)*\frac{1}{n-i}*(i+1) > \mathrm{abs}(x) \tag{9.45}$$

$$\frac{1}{2}*2*(i+1) > \mathrm{abs}(x) \tag{9.46}$$

Strict stability is determined by the two initial terms.

$$\text{if}\,(\text{abs}(x) < 1)\,\text{then}\,(w[0](\vec{x})\ \text{ is strictly stable}) \tag{9.47}$$

The denominator is non-zero within the domain of strict stability and therefore the rational function is defined within that domain.

Numerator and denominator are polynomials that are mirror inverted from the negative to the positive end. The value of the numerator at the positive bound is greater than that of the denominator. Therefore the numerator is monotonic increasing and the denominator is monotonic decreasing within the domain of strict stability. The quotient of the two is monotonic increasing.

A law of exponents applies for positions less than unity since terms of small orders dominate terms of higher orders. The radius of self-accuracy is determined by the condition on the zeroth term (9.37).

$$w[0]^k \approx \left(\mathsf{e}^{\vec{h}}\right)^k = \mathsf{e}^{k*\vec{h}} = \mathsf{e}^{\vec{x}}; \qquad \text{abs}\left(k*\vec{h}\right) \lessapprox 0.810; \qquad \text{abs}\left(\vec{h}^m\right) > \text{abs}\left(\vec{h}^{m+1}\right) \tag{9.48}$$

9.5 Exponential Function of Single Precision

An exponential function of single precision according to IEEE 754 is determined. It may be implemented according to listing 9.1 on page 76. A doubly restrained extrapolation of seven differential conditions is determined according to (9.41).

$$f\left(\vec{h}\right) = \frac{\sum^{0\le i\le 3}\left\{\frac{(6-i)!}{(3-i)!*i!}*\vec{h}^i\right\}}{\sum^{0\le i\le 3}\left\{(-1)^i*\frac{(6-i)!}{(3-i)!*i!}*\vec{h}^i\right\}} = \frac{120+60*\vec{h}+12*\vec{h}^2+\vec{h}^3}{120-60*\vec{h}+12*\vec{h}^2-\vec{h}^3} \tag{9.49}$$

The value is computed by law of exponents with $h = 0.1$.

$$f(0.1)^k \approx \mathsf{e}^{k*0.1} = \mathsf{e}^{\vec{x}} \tag{9.50}$$

The polynomial division by smallest orders is determined in order to estimate the maximum error.

$$f\left(\vec{h}\right) = \sum^{0\le i<7}\left\{\frac{\vec{h}^i}{i!}\right\} + \frac{\vec{h}^7}{4800} + \frac{\vec{h}^8}{28800} + \mathcal{O}\left(\vec{h}^9\right) \tag{9.51}$$

$$= \sum^{0\le i<7}\left\{\frac{\vec{h}^i}{i!}\right\} + \frac{\vec{h}^7}{5040} + \frac{\vec{h}^7}{100800} + \frac{\vec{h}^8}{28800} + \mathcal{O}\left(\vec{h}^9\right) \tag{9.52}$$

The maximum error is estimated by the remainder compared to the single extrapolation.

$$e\left(\vec{h}\right) = \text{abs}\left(\frac{2*\vec{h}^7}{100800}\right) + \text{abs}\left(\frac{2*\vec{h}^8}{28800}\right); \qquad e(0.1) < 2.7*10^{-12} \tag{9.53}$$

The range of single precision is about $\pm 3.403*10^{38}$ with seven significant leading digits. The domain of the extrapolation is determined.

$$\text{abs}(\vec{x}) = \log[\mathsf{e}]\left(3.403*10^{38}\right) < 90 \tag{9.54}$$

Factor k is separated into a binary number. A maximum of nine multiplications are required for the domain of single precision and a step h.

$$90 = 900 * 0.1 < 1024 * 0.1 = 2^{10} * 0.1 \tag{9.55}$$

The precision of computers is finite and usually half a bit of precision is lost for each multiplication. A maximum of two multiplications is required for each binary part. Therefore a maximum of four bits of precision is lost if double precision is used for computation.

$$\log[2]\left(2 * 9 * \frac{1}{2}\right) < 4 \tag{9.56}$$

9.6 Natural Powers of the Exponential Function

Natural powers of the exponential functions may be approximated with additional exponential conditions between origin and extrapolation point. These operators do not improve accuracy and a general expression is unknown. A few examples are given.

Exponential Square

The square of the exponential function is extrapolated. A Dirichlet condition is determined at the origin. An exponential condition is determined at three points the origin, the extrapolation point and at the center of the two. The base polynomials are determined by a system of linear equations.

$$\begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 1-\vec{x} & 1-2*\vec{x} \\ 0 & 0 & 2*\vec{x}-\vec{x}^2 & 4*\vec{x}-4*\vec{x}^2 \\ 0 & 0 & 3*\vec{x}^2-\vec{x}^3 & 12\vec{x}^2-8*\vec{x}^3 \end{bmatrix} * \begin{bmatrix} w[0](\vec{x}) \\ w[1](\vec{x}) \\ w[2](\vec{x}) \\ w[3](\vec{x}) \end{bmatrix} = \begin{bmatrix} 1 \\ \vec{x} \\ \vec{x}^2 \\ \vec{x}^3 \end{bmatrix} \tag{9.57}$$

The extrapolation is determined by the zeroth base polynomial.

$$\mathsf{e}^{2*\vec{x}} \approx \frac{3+3*\vec{x}+\vec{x}^2}{3-3*\vec{x}+\vec{x}^2}; \qquad \text{abs}(x) < \frac{1}{2} \tag{9.58}$$

The square of the exponential function is extrapolated. A Dirichlet condition is determined at the origin. The two initial exponential conditions are determined at three points the origin, the extrapolation point and at the center of the two. The base polynomials are determined by a system of linear equations.

$$\left[\begin{array}{ccccccc|c} 1 & -1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1-\vec{x} & -1 & 1-2*\vec{x} & -2 & \vec{x} \\ 0 & 0 & 2 & 2*\vec{x}-\vec{x}^2 & 2-2*\vec{x} & 4*\vec{x}-4*\vec{x}^2 & 4-8*\vec{x} & \vec{x}^2 \\ 0 & 0 & 0 & 3*\vec{x}^2-\vec{x}^3 & 6*\vec{x}-3*\vec{x}^2 & 12*\vec{x}^2-8*\vec{x}^3 & 24*\vec{x}-24\vec{x}^2 & \vec{x}^3 \\ 0 & 0 & 0 & 4*\vec{x}^3-\vec{x}^4 & 12*\vec{x}^2-4*\vec{x}^3 & 32*\vec{x}^3-16*\vec{x}^4 & 96*\vec{x}^2-64*\vec{x}^3 & \vec{x}^4 \\ 0 & 0 & 0 & 5*\vec{x}^4-\vec{x}^5 & 20*\vec{x}^3-5*\vec{x}^4 & 80*\vec{x}^4-32*\vec{x}^5 & 320*\vec{x}^3-160*\vec{x}^4 & \vec{x}^5 \\ 0 & 0 & 0 & 6*\vec{x}^5-\vec{x}^6 & 30*\vec{x}^4-6*\vec{x}^5 & 192\vec{x}^5-64*\vec{x}^6 & 960*\vec{x}^4-384*\vec{x}^5 & \vec{x}^6 \end{array}\right] \tag{9.59}$$

The extrapolation is determined by the zeroth base polynomial.

$$\mathsf{e}^{2*\vec{x}} \approx \frac{90+90*\vec{x}+39*\vec{x}^2+9*\vec{x}^3+\vec{x}^4}{90-90*\vec{x}+39*\vec{x}^2-9*\vec{x}^3+\vec{x}^4}; \qquad \text{abs}(x) < \frac{1}{2} \tag{9.60}$$

Listing 9.1: e-function of single precision in C

```
#include <math.h>
#include <stdio.h>
#include <stdlib.h>

static double wexp1n3(double const x)
{
  double const xx = x*x;
  double const A = 120.l + 12.l*xx;
  double const B = x*(60.l + xx);
  return (A+B)/(A-B);
}

double exp1(double const x)
{
  unsigned j, i; // unsigned suffices for h=0.1 and LDBL_MAX
  double wj, factor;
  // compute exponent and initial factor............................
  j = (unsigned)(fabs(x)/0.1l) + 1; // |x|/max(h)
  factor = wexp1n3(x/j); // Gewicht von x/j
  // compute power ...........................................
  wj = j&1 ? factor : 1.l; // begin with w^1 or w^0
  for(i = 2; i <= j; i <<= 1) // all exponents 2,4,8,16 <=j
  {
    factor *= factor; // w^i
    if(j&i) // if i is part of j
    {
      wj *= factor;
    }
  }
  return wj;
}

int main(int argc, char ** argv)
{
  double x, e, en;
  if(argc != 2)
  {
    fprintf(stderr, "%s x\n", argv[0]);
    exit(1);
  }
  x = atof(argv[1]);
  en = exp1(x);
  e = exp(x);
  printf("expn(%lf)=%.20lg\n", x, exp1(x));
  printf("exp (%lf)=%.20lg\n", x, exp(x));
  printf("fehler~%lg\n", (en-e)/e);
  return 0;
}
```

Exponential Cubic

The cubic of the exponential function is extrapolated. A Dirichlet condition is determined at the origin. An exponential condition is determined at four equidistant points with the origin to the left and the extrapolation point to the right end. The base polynomials are determined by a system of linear equations.

$$\begin{bmatrix} 1 & -1 & -1 & -1 & -1 \\ 0 & 1 & 1-\vec{x} & 1-2*\vec{x} & 1-3*\vec{x} \\ 0 & 0 & 2*\vec{x}-\vec{x}^2 & 4*\vec{x}-4\vec{x}^2 & 6*\vec{x}-9*\vec{x}^2 \\ 0 & 0 & 3\vec{x}^2-\vec{x}^3 & 12*\vec{x}^2-8*\vec{x}^3 & 27*\vec{x}^2-27*\vec{x}^3 \\ 0 & 0 & 4*\vec{x}^3-\vec{x}^4 & 32*\vec{x}^3-16*\vec{x}^4 & 108*\vec{x}^3-81*\vec{x}^4 \end{bmatrix} \begin{bmatrix} w[0](\vec{x}) \\ w[1](\vec{x}) \\ w[2](\vec{x}) \\ w[3](\vec{x}) \\ w[4](\vec{x}) \end{bmatrix} = \begin{bmatrix} 1 \\ \vec{x} \\ \vec{x}^2 \\ \vec{x}^3 \\ \vec{x}^4 \end{bmatrix} \tag{9.61}$$

The extrapolation is determined by the zeroth base polynomial.

$$\mathsf{e}^{3*\vec{x}} \approx \frac{12+18*\vec{x}+11*\vec{x}^2+3*\vec{x}^3}{12-18*\vec{x}+11*\vec{x}^2-3*\vec{x}^3}; \qquad \mathrm{abs}(\vec{x}) < \frac{1}{3} \tag{9.62}$$

Exponential Quartic

The quartic of the exponential function is extrapolated. A Dirichlet condition is determined at the origin. An exponential condition is determined at five equidistant points with the origin to the left and the extrapolation point to the right end. The base polynomials are determined by a system of linear equations.

$$\left[\begin{array}{cccccc|c} 1 & -1 & -1 & -1 & -1 & -1 & 1 \\ 0 & 1 & 1-\vec{x} & 1-2*\vec{x} & 1-3*\vec{x} & 1-4*\vec{x} & \vec{x} \\ 0 & 0 & 2*\vec{x}-\vec{x}^2 & 4*\vec{x}-4*\vec{x}^2 & 6*\vec{x}-9*\vec{x}^2 & 8*\vec{x}-16*\vec{x}^2 & \vec{x}^2 \\ 0 & 0 & 3*\vec{x}^2-\vec{x}^3 & 12*\vec{x}^2-8*\vec{x}^3 & 27*\vec{x}^2-27*\vec{x}^3 & 48*\vec{x}^2-64*\vec{x}^3 & \vec{x}^3 \\ 0 & 0 & 4*\vec{x}^3-\vec{x}^4 & 32*\vec{x}^3-16*\vec{x}^4 & 108*\vec{x}^3-81*\vec{x}^4 & 256*\vec{x}^3-256*\vec{x}^4 & \vec{x}^4 \\ 0 & 0 & 5*\vec{x}^4-\vec{x}^5 & 80*\vec{x}^4-32*\vec{x}^5 & 405*\vec{x}^4-243*\vec{x}^5 & 1280*\vec{x}^4-1024*\vec{x}^5 & \vec{x}^5 \end{array}\right] \tag{9.63}$$

The extrapolation is determined by the zeroth base polynomial.

$$\mathsf{e}^{4*\vec{x}} \approx \frac{60+120*\vec{x}+105*\vec{x}^2+50*\vec{x}^3+12*\vec{x}^4}{60-120*\vec{x}+105*\vec{x}^2-50*\vec{x}^3+12*\vec{x}^4}; \qquad \mathrm{abs}(\vec{x}) < \frac{1}{4} \tag{9.64}$$

Chapter 10

Sine Methods

10.1 Sine Series

The sine is a theorem (10.75) and contained by solutions to many differential problems. Approximations to the sine are discussed. These approximations are determined by a number of continuous differential equations. Any two derivatives that differ by four orders are equal.

$$\partial\left(\langle k\rangle\,;\sin(\vec{x})\,;\vec{u}\right)+\partial\left(\langle k+2\rangle\,;\sin(\vec{x})\,;\vec{u}\right)=0 \tag{10.1}$$

A number of continuous differential equations is determined at the local origin. The condition of zero order contains the required Dirichlet condition.

$$\sum^{0\le k<n}\langle\partial\left(\langle k\rangle\right);\sin(\vec{x})\,;\mathbf{0})+\partial\left(\langle k+2\rangle\,;\sin(\vec{x})\,;\mathbf{0}\right)=0\rangle \tag{10.2}$$

The polynomial coefficients are all zero since the conditions have sources of Zero.

$$\begin{bmatrix} 0! & 0 & 2! & 0 & 0 & 0 & 0 & ... \\ 0 & 1! & 0 & 3! & 0 & 0 & 0 & ... \\ 0 & 0 & 2! & 0 & 4! & 0 & 0 & ... \\ 0 & 0 & 0 & 3! & 0 & 5! & 0 & ... \\ 0 & 0 & 0 & 0 & 4! & 0 & 5! & ... \\ 0 & 0 & 0 & 0 & 0 & 5! & 0 & ... \\ 0 & 0 & 0 & 0 & 0 & 0 & 6! & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} * \begin{bmatrix} a[0] \\ a[1] \\ a[2] \\ a[3] \\ a[4] \\ a[5] \\ a[6] \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \tag{10.3}$$

The polynomial exists since the determinant is non-zero. The determinant equals a super faculty and contains at least all primes up to n.

$$\det(G)=\prod^{0\le i<n}\{i!\} \tag{10.4}$$

The tranposed polynomial is determined by weights or base polynomials $w[k](\vec{x})$.

$$f(\vec{x})=\sum^{0\le k<n}\{w[k](\vec{x})*\left(\partial\left(\langle k\rangle\,;\sin(\vec{x})\,;\mathbf{0}\right)+\partial\left(\langle k+2\rangle\,;\sin(\vec{x})\,;\mathbf{0}\right)\right)\}=0 \tag{10.5}$$

The base polynomials are determined by a system of linear equations.

$$\begin{bmatrix} 0! & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1! & 0 & 0 & 0 & 0 & 0 & \dots \\ 2! & 0 & 2! & 0 & 0 & 0 & 0 & \dots \\ 0 & 3! & 0 & 3! & 0 & 0 & 0 & \dots \\ 0 & 0 & 4! & 0 & 4! & 0 & 0 & \dots \\ 0 & 0 & 0 & 5! & 0 & 5! & 0 & \dots \\ 0 & 0 & 0 & 0 & 6! & 0 & 6! & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} * \begin{bmatrix} w[0](\vec{x}) \\ w[1](\vec{x}) \\ w[2](\vec{x}) \\ w[3](\vec{x}) \\ w[4](\vec{x}) \\ w[5](\vec{x}) \\ w[6](\vec{x}) \\ \vdots \end{bmatrix} = \begin{bmatrix} \vec{x}^0 \\ \vec{x}^1 \\ \vec{x}^2 \\ \vec{x}^3 \\ \vec{x}^4 \\ \vec{x}^5 \\ \vec{x}^6 \\ \vdots \end{bmatrix} \tag{10.6}$$

The system is triangular such that the solution is explicitly available. The solution develops to finite series of $\sin(\vec{x})$ and $\cos(\vec{x})$. Therefore the base polynomials equal the derivatives in reverse differential order. The system may be separated into two triangular systems since each weight only depends on the pre-predecessor.

$$w[0](\vec{x}) = \frac{\vec{x}^0}{0!} = \cos[1](\vec{x}) \tag{10.7}$$

$$w[1](\vec{x}) = \frac{\vec{x}^1}{1!} = \sin[1](\vec{x}) \tag{10.8}$$

$$w[2](\vec{x}) = \frac{1}{2!} * \left(\vec{x}^2 - 2! * w[0](\vec{x})\right) = -\frac{\vec{x}^0}{0!} + \frac{\vec{x}^2}{2!} = -\cos[2](\vec{x}) \tag{10.9}$$

$$w[3](\vec{x}) = \frac{1}{3!} * \left(\vec{x}^3 - 3! * w[1](\vec{x})\right) = -\frac{\vec{x}^1}{1!} + \frac{\vec{x}^3}{3!} = -\sin[2](\vec{x}) \tag{10.10}$$

$$w[4](\vec{x}) = \frac{1}{4!} * \left(\vec{x}^4 - 4! * w[2](\vec{x})\right) = \frac{\vec{x}^0}{0!} - \frac{\vec{x}^2}{2!} + \frac{\vec{x}^4}{4!} = \cos[3](\vec{x}) \tag{10.11}$$

$$w[5](\vec{x}) = \frac{1}{5!} * \left(\vec{x}^5 - 5! * w[3](\vec{x})\right) = \frac{\vec{x}^1}{1!} - \frac{\vec{x}^3}{3!} + \frac{\vec{x}^5}{5!} = \sin[3](\vec{x}) \tag{10.12}$$

$$w[6](\vec{x}) = \frac{1}{6!} * \left(\vec{x}^6 - 6! * w[4](\vec{x})\right) = -\frac{\vec{x}^0}{0!} + \frac{\vec{x}^2}{2!} - \frac{\vec{x}^4}{4!} + \frac{\vec{x}^6}{6!} = -\cos[4](\vec{x}) \tag{10.13}$$

$$w[j](\vec{x}) = \sum^{h \le i < j/2} \left\{(-1)^{k+i} * \frac{\vec{x}^{2*i+h}}{(2*i+h)!}\right\}; \qquad \begin{array}{l} h = \mathrm{mod}\,(j;2) \\ k = \mathrm{mod}\,(j;4)\,/2 \end{array} \tag{10.14}$$

Approximations to the sine and cosine are defined by a finite series within the domain of strict stabilty.

$$\sin(\vec{x}) \approx \sum^{0 \le i < m} \left\{(-1)^i * \frac{\vec{x}^{2*i+1}}{(2*i+1)!}\right\} = \vec{x} * \left(1 - \frac{\vec{x}^2}{2*3} * \left(1 - \frac{\vec{x}^2}{4*5} * (...)\right)\right); \qquad \mathrm{abs}(\vec{x}) < \sqrt{3} \tag{10.15}$$

$$\cos(\vec{x}) \approx \sum^{0 \le i < m} \left\{(-1)^i * \frac{\vec{x}^{2*i}}{(2*i)!}\right\} = 1 - \frac{\vec{x}^2}{1*2} * \left(1 - \frac{\vec{x}^2}{3*4} * \left(1 - \frac{\vec{x}^2}{5*6} * (...)\right)\right); \quad \mathrm{abs}(\vec{x}) < 1 \tag{10.16}$$

The sine is a regular undulating curve and the cosine a shifted sine.

$$\sin(\vec{x}) = \sin(\vec{x} + 2 * j * \pi) \tag{10.17}$$

$$\cos(\vec{x}) = \sin\left(\vec{x} - \frac{\pi}{2}\right) \tag{10.18}$$

The values of the sine repeat above $^{\pi}/_{2}$.

$$\sin(\vec{x}) = \sin\left(\frac{\pi}{2} - \vec{x}\right); \qquad \frac{1}{2} * \pi \leq \vec{x} \leq \pi \tag{10.19}$$

$$\sin(\vec{x}) = -\sin\left(\frac{\pi}{2} - \vec{x}\right); \qquad \pi \leq \vec{x} \leq \frac{3}{2} * \pi \tag{10.20}$$

$$\sin(\vec{x}) = -\sin(\vec{x}); \qquad \frac{3}{2} * \pi \leq \vec{x} \leq 2 * \pi \tag{10.21}$$

The domain of strict stability of the sine is larger than the domain of unique values such that the series suffices as approximation on the entire domain of the sine.

$$\frac{1}{2} * \pi < \sqrt{3} \tag{10.22}$$

10.2 Hyperbolic Sine Series

The hyperbolic sine and cosine are defined by the exponential function.

$$\sinh(\vec{x}) = \frac{1}{2} * (\exp(\vec{x}) - \exp(-\vec{x})) \tag{10.23}$$

$$\cosh(\vec{x}) = \frac{1}{2} * (\exp(\vec{x}) + \exp(-\vec{x})) \tag{10.24}$$

Approximations are determined by a number of continuous differential equations. Any two derivatives that differ by two orders are equal.

$$\partial\left(\langle k\rangle ; \sin(\vec{x}) ; \vec{u}\right) - \partial\left(\langle k+2\rangle ; \sin(\vec{x}) ; \vec{u}\right) = 0 \tag{10.25}$$

The tranposed polynomial is determined by weights or base polynomials $w[k](\vec{x})$.

$$f(\vec{x}) = \sum^{0\leq k<n} \left\{ w[k](\vec{x}) * \left(\partial\left(\langle k\rangle ; \sin(\vec{x}) ; \mathbf{0}\right) - \partial\left(\langle k+2\rangle ; \sin(\vec{x}) ; \mathbf{0}\right)\right)\right\} = 0 \tag{10.26}$$

The base polynomials are determined by a system of linear equations.

$$\begin{bmatrix} 0! & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1! & 0 & 0 & 0 & 0 & 0 & \dots \\ -2! & 0 & 2! & 0 & 0 & 0 & 0 & \dots \\ 0 & -3! & 0 & 3! & 0 & 0 & 0 & \dots \\ 0 & 0 & -4! & 0 & 4! & 0 & 0 & \dots \\ 0 & 0 & 0 & -5! & 0 & 5! & 0 & \dots \\ 0 & 0 & 0 & 0 & -6! & 0 & 6! & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} * \begin{bmatrix} w[0](\vec{x}) \\ w[1](\vec{x}) \\ w[2](\vec{x}) \\ w[3](\vec{x}) \\ w[4](\vec{x}) \\ w[5](\vec{x}) \\ w[6](\vec{x}) \\ \vdots \end{bmatrix} = \begin{bmatrix} \vec{x}^0 \\ \vec{x}^1 \\ \vec{x}^2 \\ \vec{x}^3 \\ \vec{x}^4 \\ \vec{x}^5 \\ \vec{x}^6 \\ \vdots \end{bmatrix} \tag{10.27}$$

The system is triangular such that the solution is explicitly available. The solution develops to finite series of $\sinh(\vec{x})$ and $\cosh(\vec{x})$. Therefore the base polynomials equal the derivatives in reverse differential order. The system may be separated into two triangular systems since each weight only depends on the pre-predecessor.

$$w[0](\vec{x}) = \frac{\vec{x}^0}{0!} = \cosh[1](\vec{x}) \tag{10.28}$$
$$w[1](\vec{x}) = \frac{\vec{x}^1}{1!} = \sinh[1](\vec{x}) \tag{10.29}$$
$$w[2](\vec{x}) = \frac{1}{2!} * \left(\vec{x}^2 + 2! * w[0](\vec{x})\right) = \frac{\vec{x}^0}{0!} + \frac{\vec{x}^2}{2!} = \cosh[2](\vec{x}) \tag{10.30}$$
$$w[3](\vec{x}) = \frac{1}{3!} * \left(\vec{x}^3 + 3! * w[1](\vec{x})\right) = \frac{\vec{x}^1}{1!} + \frac{\vec{x}^3}{3!} = \sinh[2](\vec{x}) \tag{10.31}$$
$$w[4](\vec{x}) = \frac{1}{4!} * \left(\vec{x}^4 + 4! * w[2](\vec{x})\right) = \frac{\vec{x}^0}{0!} + \frac{\vec{x}^2}{2!} + \frac{\vec{x}^4}{4!} = \cosh[3](\vec{x}) \tag{10.32}$$
$$w[5](\vec{x}) = \frac{1}{5!} * \left(\vec{x}^5 + 5! * w[3](\vec{x})\right) = \frac{\vec{x}^1}{1!} + \frac{\vec{x}^3}{3!} + \frac{\vec{x}^5}{5!} = \sinh[3](\vec{x}) \tag{10.33}$$
$$w[6](\vec{x}) = \frac{1}{6!} * \left(\vec{x}^6 + 6! * w[4](\vec{x})\right) = \frac{\vec{x}^0}{0!} + \frac{\vec{x}^2}{2!} + \frac{\vec{x}^4}{4!} + \frac{\vec{x}^6}{6!} = \cosh[4](\vec{x}) \tag{10.34}$$
$$w[j](\vec{x}) = \sum^{h \le i < j/2} \left\{ \frac{\vec{x}^{2*i+h}}{(2*i+h)!} \right\}; \qquad h = \mathrm{mod}\,(j;2) \tag{10.35}$$

Approximations to the hyperbolic sine and cosine are defined by a finite series within the domain of strict stabilty.

$$\sinh(\vec{x}) \approx \sum^{0 \le i < m} \left\{ \frac{\vec{x}^{2*i+1}}{(2*i+1)!} \right\} = \vec{x} * \left(1 + \frac{\vec{x}^2}{2*3} * \left(1 + \frac{\vec{x}^2}{4*5} * (...)\right)\right); \qquad \mathrm{abs}(\vec{x}) < \sqrt{3} \tag{10.36}$$
$$\cosh(\vec{x}) \approx \sum^{0 \le i < m} \left\{ \frac{\vec{x}^{2*i}}{(2*i)!} \right\} = 1 + \frac{\vec{x}^2}{1*2} * \left(1 + \frac{\vec{x}^2}{3*4} * \left(1 + \frac{\vec{x}^2}{5*6} * (...)\right)\right); \quad \mathrm{abs}(\vec{x}) < 1 \tag{10.37}$$

Approximation outside the domain of strict stability may be computed with approximations to the exponential functions.

10.3 Sine Repetition

The sine repetition is a numerical pattern that determines a number of analytical functions. The values of a function are repeatedly determined by two preceding values. These values are scaled by the same weights due to a uniform discretization.

$$y\left[\vec{h}\right] = y[L] * w[L] + y[0] * w[0] \tag{10.38}$$
$$y\left[2*\vec{h}\right] = y[0] * w[L] + y\left[\vec{h}\right] * w[0] \tag{10.39}$$
$$= y[0] * w[L] + (y[L] * w[L] + y[0] * w[0]) * w[0] \tag{10.40}$$
$$= y[L] * w[L] * w[0] + y[0] * \left(w[L] + w[0]^2\right) \tag{10.41}$$
$$y\left[3*\vec{h}\right] = y\left[\vec{h}\right] * w[L] + y\left[2*\vec{h}\right] * w[0] \tag{10.42}$$
$$= y[L] * \left(w[L]^2 + w[L] * w[0]^2\right) + y[0] * \left(2 * w[L] * w[0] + w[0]^3\right) \tag{10.43}$$
$$...$$

The repetition contains composed weights that are defined by a method of four arguments. The sum is similar to a binomial expansion but does not reduce to a basic operation.

$$\mathrm{scw}(w[L]\,;w[0]\,;j;k) = \sum^{0\le i\le \frac{j-k}{2}} \left\{ \binom{j-k-i}{i} * w[L]^{i+k} * w[0]^{j-k-2*i} \right\} \tag{10.44}$$

The j-th value is determined.

$$y\left[j*\vec{h}\right] = y[L]*\mathrm{scw}(w[L]\,;w[L]\,;j;1) + y[0]*\mathrm{scw}(w[L]\,;w[0]\,;j;0) \tag{10.45}$$

The Fibonacci repetition is a special case of the sine repetition.

$$F_{j+2} = F_{j+1} + F_j \tag{10.46}$$

A Fibonacci number F is determined.

$$F_j = \mathrm{scw}(1;1;j;1) = \sum^{0\le i\le \frac{j-1}{2}} \left\{ \binom{j-1-i}{i} \right\} \tag{10.47}$$

10.4 Tridiagonal System of Equations

A uniform tridiagonal square system of equations is determined by n equidistant base points. The bounds of the domain are contained in the sources.

$$\begin{bmatrix} B & C & & & & & \\ A & B & C & & & & \\ & A & B & C & & & \\ & & & ... & & & \\ & & & & A & B & C \\ & & & & & A & B & C \\ & & & & & & A & B \end{bmatrix} * \begin{bmatrix} y[1] \\ y[2] \\ y[3] \\ . \\ y[n-4] \\ y[n-3] \\ y[n-2] \end{bmatrix} = \begin{bmatrix} q[1] \\ q[2] \\ q[3] \\ . \\ q[n-4] \\ q[n-3] \\ q[n-2] \end{bmatrix} \tag{10.48}$$

The determinants of a base matrix on a domain of two and three base points are defined.

$$D[1] = \det\big([1]\big) =_{\mathrm{def}} 1 \tag{10.49}$$

$$D[2] = \det\big([B]\big) =_{\mathrm{def}} B \tag{10.50}$$

A determinant of a base matrix of four or more equations is defined by the next two smaller determinants.

$$D[j+2] = B*D[j+1] - A*C*D[j]\,; \qquad j \ge 1 \tag{10.51}$$

$$D[3] = \det\left(\begin{bmatrix} B & C \\ A & B \end{bmatrix}\right) = B*B - A*C*1 = B^2 - A*C \tag{10.52}$$

$$D[4] = \det\left(\begin{bmatrix} B & C & 0 \\ A & B & C \\ 0 & A & B \end{bmatrix}\right) = B*\big(B^2 - A*C\big) - B = B^3 - 2*A*B*C \tag{10.53}$$

$$D[5] = \det\left(\begin{bmatrix} B & C & 0 & 0 \\ A & B & C & 0 \\ 0 & A & B & C \\ 0 & 0 & A & B \end{bmatrix}\right) = B^4 - 3*A*B^2*C + A^2*C^2 \tag{10.54}$$

A determinant is defined by a pattern.

$$D[j] = \sum^{0 \le i \le \frac{j-1}{2}} \left\{ (-1)^i * \binom{j-1-i}{i} * A^i * B^{j-1-2*i} * C^i \right\}; \qquad j \ge 1 \tag{10.55}$$

A source matrix of $n-2$ dimensions and the k-th column replaced is determined.

$$Q[n-1][k] = \begin{bmatrix} B & C & & & q[1] & & & & \\ A & B & C & & q[2] & & & & \\ & A & B & C & q[3] & & & & \\ & & & . & . & . & & & \\ & & & & q[n-4] & A & B & C & \\ & & & & q[n-3] & & A & B & A \\ & & & & q[n-2] & & & A & B \end{bmatrix} \tag{10.56}$$

The determinant of a source matrix of column k is defined by a sum above the k-th row and a sum below and on the k-th row.

$$\begin{aligned} \det(Q[n-1][k]) = \; & D[n-1-k] * \sum^{0 \le i < k} \left\{ (-1)^{k+i} * A^{k-i} * q[i+1] * D[i+1] \right\} \\ & + D[k+1] * \sum^{k \le i < n-1} \left\{ (-1)^{k+i} * C^{i-k} * q[i+1] * D[n-1-i] \right\} \end{aligned} \tag{10.57}$$

The solution is determined by Cramer's rule.

$$\sum^{0 \le k < n-2} \left\langle y[k+1] = \frac{\det(Q[n-1][k])}{D[n-1]} \right\rangle \tag{10.58}$$

10.5 Sine Extrapolation

The sine extrapolation method is a repeated uniform extrapolation of a sine extrapolation operator and results the sine theorem.

The simplest sine extrapolation operator is determined by two Dirichlet conditions left to and at the origin and one condition of simple harmonic motion of a distribution coefficient c at the origin.

$$\partial f[\langle 0 \rangle]\left(-\vec{h}\right) = y[L] \tag{10.59}$$

$$\partial f[\langle 0 \rangle]\left(\vec{\mathbf{0}}\right) = y[0] \tag{10.60}$$

$$c^2 * \partial f[\langle 0 \rangle]\left(\vec{\mathbf{0}}\right) + \partial f[\langle 2 \rangle]\left(\vec{\mathbf{0}}\right) = 0 \tag{10.61}$$

The transposed polynomial of $\vec{h}$ is determined by a system of linear equations if the two points are not coincident.

$$\begin{bmatrix} 1 & 1 & c^2 \\ -\vec{h} & 0 & 0 \\ \vec{h}^2 & 0 & 2 \end{bmatrix} * \begin{bmatrix} w[L] \\ w[0] \\ w[1] \end{bmatrix} = \begin{bmatrix} 1 \\ \vec{h} \\ \vec{h}^2 \end{bmatrix}; \qquad \begin{aligned} w[L] &= -1 \\ w[0] &= 2 - c^2 * \vec{h}^2 \\ w[1] &= \vec{h}^2 \end{aligned} \tag{10.62}$$

The value of the transposed polynomial of $\vec{h}$ is determined.

$$y\left[\vec{h}\right] = y[L] * w[L] + y[0] * w[0] \tag{10.63}$$

Suppose the solution is a sine.

$$f(\vec{x}) = R * \sin(\varphi + d * \vec{x}) \tag{10.64}$$

The values of the sine and base polynomials are substituted.

$$R * \sin\left(\varphi + d * \vec{h}\right) = R * \sin\left(\varphi - d * \vec{h}\right) * (-1) + R * \sin(\varphi) * \left(2 - c^2 * \vec{h}^2\right) \tag{10.65}$$

A trigonometric addition formula applies.

$$\sin(\vec{x}[0] \pm \vec{x}[1]) = \sin(\vec{x}[0]) * \cos(\vec{x}[1]) \pm \cos(\vec{x}[0]) * \sin(\vec{x}[1]) \tag{10.66}$$

The formula is applied and two terms cancel. The scalar $R*\sin(\varphi)$ cancels. Note that $w[L]$ equals negative One.

$$\cos\left(d * \vec{h}\right) = -\cos\left(d * \vec{h}\right) + \left(2 - c^2 * \vec{h}^2\right) \tag{10.67}$$

The approximation does not exist if the difference $\vec{h}$ equals Zero. The domain of the difference is positive per definition. The analytical solution is approximated by (10.16) and tends to the numeric solution if the difference $\vec{h}$ tends to its lower bound.

$$c^2 * \vec{h}^2 = 2 - 2 * \cos\left(d * \vec{h}\right) = 2 - 2 * \left(1 - \frac{d^2 * \vec{h}^2}{2!} + \frac{d^4 * \vec{h}^4}{4!} + \mathcal{O}\left(\vec{h}^6\right)\right) \tag{10.68}$$

$$c^2 = d^2 + 2 * \frac{d^4 * \vec{h}^2}{4!} - \mathcal{O}\left(\vec{h}^2\right); \qquad \lim_{\vec{h} \to \vec{\mathbf{0}}} \left(2 * \frac{d^4 * \vec{h}^2}{4!} - \mathcal{O}\left(\vec{h}^4\right)\right) = 0 \tag{10.69}$$

The upper bound of difference $\vec{h}$ is determined by the domain of the $\arccos\left(\vec{h}\right)$.

$$d * \vec{h} = \arccos\left(1 - \frac{1}{2} * c^2 * \vec{h}^2\right); \qquad \text{abs}\left(1 - \frac{1}{2} * c^2 * \vec{h}^2\right) \le 1; \qquad c^2 * \vec{h}^2 \le 2 \tag{10.70}$$

The value of arccos(1) is Zero and would result a difference of Zero. Therefore the upper bound is excluded. The intersected domain is determined.

$$0 < \vec{h} < \frac{\sqrt{2}}{c} \tag{10.71}$$

The extrapolation is repeated by the same operator according to equation 10.63 and section 10.3.

The repeated extrapolation is a sum of two sines. The constant ϕ is an unknown non-zero offset.

$$\mathrm{scw}(w[L]\,;w[0]\,;j;1) = \frac{\sin\left(d * j * \vec{h}\right)}{\sin\left(-d * \vec{h}\right)} \tag{10.72}$$

$$\mathrm{scw}(w[L]\,;w[0]\,;j;0) = \frac{\sin\left(d * j * \vec{h} + \phi\right)}{\sin\left(-d * \vec{h}\right)} \tag{10.73}$$

$$y\left[j * \vec{h}\right] = \frac{y[L] * \sin\left(d * j * \vec{h}\right) + y[0] * \sin\left(d * j * \vec{h} + \phi\right)}{\sin\left(-d * \vec{h}\right)} \tag{10.74}$$

The sine theorem is determined by the left composed weight only since the value at the origin $y[0]$ of the sine equals Zero. The signs are rearranged. An implementation is presented with listing 10.1.

$$w[0] = 2 - c^2 * \vec{h}^2; \quad d * \vec{h} = \arccos\left(1 - \frac{1}{2} * c^2 * \vec{h}^2\right); \quad 0 < \vec{h} < \frac{\sqrt{2}}{c} \tag{10.75a}$$

$$\frac{\sin\left(d * j * \vec{h}\right)}{\sin\left(d * \vec{h}\right)} = \sum^{0 \le i \le \frac{j-1}{2}} \left\{(-1)^i * \binom{j-1-i}{i} * w[0]^{j-1-2*i}\right\} \tag{10.75b}$$

The sine method produces a local polynomial at each step.

$$y\left[\vec{h}\right] = 1 \tag{10.76}$$

$$y\left[2 * \vec{h}\right] = 2 - c^2 * \vec{h}^2 \tag{10.77}$$

$$y\left[3 * \vec{h}\right] = 3 - 4 * c^2 * \vec{h}^2 + c^4 * \vec{h}^4 \tag{10.78}$$

$$y\left[4 * \vec{h}\right] = 4 - 10 * c^2 * \vec{h}^2 + 6 * c^4 * \vec{h}^4 - c^6 * \vec{h}^6 \tag{10.79}$$

$$\ldots \tag{10.80}$$

The sine method reduces to a line if the distribution coefficient equals Zero. This is the simplest solution to Laplace equation.

$$\sum^{0 \le i \le \frac{j-1}{2}} \left\{(-1)^i * \binom{j-1-i}{i} * 2^{j-1-2*i}\right\} = j \tag{10.81}$$

10.6 Sine Interpolation

The sine interpolation method interpolates a sine on a uniformly discretized domain by a sine interpolation operator.

The simplest sine interpolation operator is determined by two Dirichlet conditions left and right to a base point or local origin and one condition of harmonic motion of a distribution coefficient

c and a source s at a base point.

$$\partial f[\langle 0\rangle]\Big(-\vec{h}\Big) = y[L] \tag{10.82}$$

$$c^2 * \partial f[\langle 0\rangle]\Big(\vec{\mathbf{0}}\Big) + \partial f[\langle 2\rangle]\Big(\vec{\mathbf{0}}\Big) = s\Big[\vec{\mathbf{0}}\Big] \tag{10.83}$$

$$\partial f[\langle 0\rangle]\Big(\vec{h}\Big) = y[R] \tag{10.84}$$

The base polynomials of the operator are determined by a system of linear equations.

$$\begin{bmatrix} 1 & c^2 & 1 \\ -\vec{h} & 0 & \vec{h} \\ \vec{h}^2 & 2 & \vec{h}^2 \end{bmatrix} * \begin{bmatrix} w[L] \\ w[C] \\ w[R] \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{10.85a}$$

$$w[L] = w[R] = \frac{1}{2 - c^2 * \vec{h}^2} = \frac{1}{W}; \qquad w[C] = -\frac{\vec{h}^2}{2 - c^2 * \vec{h}^2} \tag{10.85b}$$

The value at a local origin $\vec{\mathbf{0}}$ is determined. The domain of the difference $\vec{h}$ is determined by strict stability according to (10.71).

$$y\Big[\vec{\mathbf{0}}\Big] = \frac{1}{W} * y[L] + \frac{1}{W} * y[R] + w[C] * s\Big[\vec{\mathbf{0}}\Big] \tag{10.86}$$

An interpolation is determined by a number of polynomials joined by Dirichlet conditions.

$$y[0] = q[0] \tag{10.87a}$$

$$\sum^{1 \le i < n-1} \Big\langle -\frac{1}{W} * y[i-1] + y[i] - \frac{1}{W} * y[i+1] = w[C] * s[i] \Big\rangle \tag{10.87b}$$

$$y[n-1] = q[n-1] \tag{10.87c}$$

The equations of unknowns are normalized to the neighbours.

$$\sum^{1 \le i < n} \Big\langle -y[i-1] + W * y[i] - y[i+1] = W * w[C] * s[i] = -\vec{h}^2 * s[i] = q[i] \Big\rangle \tag{10.88}$$

A symmetric uniform tridiagonal system of linear equations determines the unknowns.

$$\begin{bmatrix} W & -1 & & & & & & \\ -1 & W & -1 & & & & & \\ & -1 & W & -1 & & & & \\ & & & . \; . \; . & & & & \\ & & & & -1 & W & -1 & \\ & & & & & -1 & W & -1 \\ & & & & & & -1 & W \end{bmatrix} * \begin{bmatrix} y[1] \\ y[2] \\ y[3] \\ . \\ y[n-4] \\ y[n-3] \\ y[n-2] \end{bmatrix} = \begin{bmatrix} q[1] \\ q[2] \\ q[3] \\ . \\ q[n-4] \\ q[n-3] \\ q[n-2] \end{bmatrix} \tag{10.89}$$

A determinant according to (10.55) results the sine theorem.

$$W = 2 - c^2 * \vec{h}^2; \qquad d * \vec{h} = \arccos\Big(1 - \frac{1}{2} * c^2 * \vec{h}^2\Big); \qquad 0 < \vec{h} < \frac{\sqrt{2}}{c} \tag{10.90a}$$

$$D[j] = \sum^{0 \le i \le \frac{j-1}{2}} \Big\{ (-1)^i * \binom{j-1-i}{i} * W^{j-1-2*i} \Big\} = \frac{\sin\Big(d * j * \vec{h}\Big)}{\sin\Big(d * \vec{h}\Big)} \tag{10.90b}$$

The determinant of a source matrix is defined according to (10.57). All signs cancel.

$$\begin{aligned}\det(Q[n-1][k]) &= D[n-1-k] * \sum^{0\le i<k} \left\{(-1)^{k+i} * (-1)^{k-i} * q[i+1] * D[i+1]\right\} \\ &\quad + D[k+1] * \sum^{k\le i<n-1} \left\{(-1)^{k+i} * (-1)^{i-k} * q[i+1] * D[n-1-i]\right\} \\ &= \frac{\sin\left(d*(n-1-k)*\vec{h}\right)}{\sin\left(d*\vec{h}\right)} * \sum^{0\le i<k} \left\{ q[i+1] * \frac{\sin\left(d*(i+1)*\vec{h}\right)}{\sin\left(d*\vec{h}\right)} \right\} \\ &\quad + \frac{\sin\left(d*(k+1)*\vec{h}\right)}{\sin\left(d*\vec{h}\right)} * \sum^{k\le i<n-1} \left\{ q[i+1] * \frac{\sin\left(d*(n-1-i)*\vec{h}\right)}{\sin\left(d*\vec{h}\right)} \right\}\end{aligned} \tag{10.91}$$

The determinant of the base matrix is defined.

$$D[n-1] = \frac{\sin\left(d*(n-1)*\vec{h}\right)}{\sin\left(d*\vec{h}\right)} \tag{10.92}$$

The solution is determined by Cramer's rule.

$$\sum^{0\le k<n-2} \left\langle y[k+1] = \frac{\sin\left(d*(n-1-k)*\vec{h}\right) * \sum^{0\le i<k} \left\{q[i+1] * \sin\left(d*(i+1)*\vec{h}\right)\right\} + \sin\left(d*(k+1)*\vec{h}\right) * \sum^{k\le i<n-1} \left\{q[i+1] * \sin\left(d*(n-1-i)*\vec{h}\right)\right\}}{\sin\left(d*(n-1)*\vec{h}\right)} \right\rangle \tag{10.93}$$

A determinant according to (10.55) results a line if the distribution coefficient c equals Zero.

$$D[j] = \sum^{0\le i\le \frac{j-1}{2}} \left\{ (-1)^i * \binom{j-1-i}{i} * 2^{j-1-2*i} \right\} = j \tag{10.94a}$$

A solution to Poisson's equation is determined.

$$\sum^{0\le k<n-2} \left\langle y[k+1] = \frac{(n-1-k) * \sum^{0\le i<k} \{q[i+1]*(i+1)\} + (k+1) * \sum^{k\le i<n-1} \{q[i+1]*(n-1-i)\}}{n-1} \right\rangle \tag{10.94b}$$

A solution to Laplace's equation is determined.

$$\sum^{0\le k<n-2} \left\langle y[k+1] = \frac{(n-1-k)*q[1] + (k+1)*q[n-2]}{n-1} \right\rangle \tag{10.94c}$$

10.7 Hyperbolic Sine Extrapolation

The hyperbolic sine extrapolation method is a repeated uniform extrapolation of a hyperbolic sine extrapolation operator and results the hyperbolic sine theorem.

The hyperbolic sine extrapolation operator is determined by two Dirichlet conditions left to and at the origin and a hyperbolic sine condition at the origin.

$$\partial f[\langle 0\rangle]\left(-\vec{h}\right) = y[L] \tag{10.95}$$

$$\partial f[\langle 0\rangle]\left(\vec{\mathbf{0}}\right) = y[0] \tag{10.96}$$

$$c^2 * \partial f[\langle 0\rangle]\left(\vec{\mathbf{0}}\right) - \partial f[\langle 2\rangle]\left(\vec{\mathbf{0}}\right) = 0 \tag{10.97}$$

The transposed polynomial of $\vec{h}$ is determined by a system of linear equations if the two points are not coincident.

$$\begin{bmatrix} 1 & 1 & c^2 \\ -\vec{h} & 0 & 0 \\ \vec{h}^2 & 0 & -2 \end{bmatrix} * \begin{bmatrix} w[L] \\ w[0] \\ w[1] \end{bmatrix} = \begin{bmatrix} 1 \\ \vec{h} \\ \vec{h}^2 \end{bmatrix}; \qquad \begin{array}{l} w[L] = -1 \\ w[0] = 2 + c^2 * \vec{h}^2 \\ w[1] = \vec{h}^2 \end{array} \tag{10.98}$$

The value of the transposed polynomial of $\vec{h}$ is determined.

$$y\left[\vec{h}\right] = y[L] * w[L] + y[0] * w[0] \tag{10.99}$$

Suppose the solution is a hyperbolic sine.

$$f(\vec{x}) = R * \sinh(\varphi + d * \vec{x}) \tag{10.100}$$

The values of the hyperbolic sine and base polynomials are substituted.

$$R * \sinh\left(\varphi + d * \vec{h}\right) = R * \sinh\left(\varphi - d * \vec{h}\right) * (-1) + R * \sinh(\varphi) * \left(2 + c^2 * \vec{h}^2\right) \tag{10.101}$$

A hyperbolic addition formula applies.

$$\sinh(\vec{x}[0] \pm \vec{x}[1]) = \sinh(\vec{x}[0]) * \cosh(\vec{x}[1]) \pm \cosh(\vec{x}[0]) * \sinh(\vec{x}[1]) \tag{10.102}$$

The formula is applied and two terms cancel. The scalar $R * \sinh(\varphi)$ cancels. Note that $w[L]$ equals negative One.

$$\cosh\left(d * \vec{h}\right) = -\cosh\left(d * \vec{h}\right) + \left(2 + c^2 * \vec{h}^2\right) \tag{10.103}$$

The approximation does not exist if the difference $\vec{h}$ equals Zero. The domain of the difference is positive per definition. The analytical solution is approximated by (10.37) and tends to the numeric solution if the difference $\vec{h}$ tends to its lower bound.

$$c^2 * \vec{h}^2 = -2 + 2 * \cosh\left(d * \vec{h}\right) = -2 + 2 * \left(1 + \frac{d^2 * \vec{h}^2}{2!} + \frac{d^4 * \vec{h}^4}{4!} + \mathcal{O}\left(\vec{h}^6\right)\right) \tag{10.104}$$

$$c^2 = d^2 + 2 * \frac{d^4 * \vec{h}^2}{4!} + \mathcal{O}\left(\vec{h}^2\right); \qquad \lim_{\vec{h} \to \vec{\mathbf{0}}} \left(2 * \frac{d^4 * \vec{h}^2}{4!} + \mathcal{O}\left(\vec{h}^4\right)\right) = 0 \tag{10.105}$$

An upper bound of difference $\vec{h}$ does not exist according to the definition of the $\operatorname{arcosh}\left(\vec{h}\right)$.

$$d * \vec{h} = \operatorname{arcosh}\left(1 + \frac{1}{2} * c^2 * \vec{h}^2\right); \qquad 1 + \frac{1}{2} * c^2 * \vec{h}^2 \geq 1; \qquad c^2 * \vec{h}^2 \geq -2 \tag{10.106}$$

The intersected domain is determined.

$$0 < \vec{h} \tag{10.107}$$

The extrapolation is repeated by the same operator according to equation 10.99 and section 10.3. The repeated extrapolation is a sum of two sines. The constant ϕ is an unknown non-zero offset.

$$\operatorname{scw}(w[L]\,; w[0]\,; j; 1) = \frac{\sinh\left(d * j * \vec{h}\right)}{\sinh\left(-d * \vec{h}\right)} \tag{10.108}$$

$$\operatorname{scw}(w[L]\,; w[0]\,; j; 0) = \frac{\sinh\left(d * j * \vec{h} + \phi\right)}{\sinh\left(-d * \vec{h}\right)} \tag{10.109}$$

$$y\left[j * \vec{h}\right] = \frac{y[L] * \sinh\left(d * j * \vec{h}\right) + y[0] * \sinh\left(d * j * \vec{h} + \phi\right)}{\sinh\left(-d * \vec{h}\right)} \tag{10.110}$$

The hyperbolic sine theorem is determined by the left composed weight only since the value at the origin $y[0]$ of the hyperbolic sine equals Zero. The signs are rearranged.

$$w[0] = 2 + c^2 * \vec{h}^2; \quad d * \vec{h} = \operatorname{arcosh}\left(1 + \frac{1}{2} * c^2 * \vec{h}^2\right); \quad 0 < \vec{h} \tag{10.111a}$$

$$\frac{\sinh\left(d * j * \vec{h}\right)}{\sinh\left(d * \vec{h}\right)} = \sum^{0 \leq i \leq \frac{j-1}{2}} \left\{(-1)^i * \binom{j-1-i}{i} * w[0]^{j-1-2*i}\right\} \tag{10.111b}$$

The hyperbolic sine method produces a local polynomial at each step.

$$y\left[\vec{h}\right] = 1 \tag{10.112}$$

$$y\left[2 * \vec{h}\right] = 2 + c^2 * \vec{h}^2 \tag{10.113}$$

$$y\left[3 * \vec{h}\right] = 3 + 4 * c^2 * \vec{h}^2 + c^4 * \vec{h}^4 \tag{10.114}$$

$$y\left[4 * \vec{h}\right] = 4 + 10 * c^2 * \vec{h}^2 + 6 * c^4 * \vec{h}^4 + c^6 * \vec{h}^6 \tag{10.115}$$

$$\ldots \tag{10.116}$$

The hyperbolic sine method reduces to the same line as the sine method if the distribution coefficient equals Zero.

$$\sum^{0 \leq i \leq \frac{j-1}{2}} \left\{(-1)^i * \binom{j-1-i}{i} * 2^{j-1-2*i}\right\} = j \tag{10.117}$$

10.8 Hyperbolic Sine Interpolation

The hyperbolic sine interpolation method interpolates a hyperbolic sine on a uniformly discretized domain by a hyperbolic sine interpolation operator.

The simplest hyperbolic sine interpolation operator is determined by two Dirichlet conditions left and right to a base point or local origin and one condition of hyperbolic harmonic motion of a distribution coefficient c and a source s at a base point.

$$\partial f[\langle 0\rangle]\left(-\vec{h}\right) = y[L] \tag{10.118}$$

$$c^2 * \partial f[\langle 0\rangle]\left(\vec{\mathbf{0}}\right) - \partial f[\langle 2\rangle]\left(\vec{\mathbf{0}}\right) = s\left[\vec{\mathbf{0}}\right] \tag{10.119}$$

$$\partial f[\langle 0\rangle]\left(\vec{h}\right) = y[R] \tag{10.120}$$

The base polynomials of the operator are determined by a system of linear equations.

$$\begin{bmatrix} 1 & c^2 & 1 \\ -\vec{h} & 0 & \vec{h} \\ \vec{h}^2 & -2 & \vec{h}^2 \end{bmatrix} * \begin{bmatrix} w[L] \\ w[C] \\ w[R] \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{10.121a}$$

$$w[L] = w[R] = \frac{1}{2 + c^2 * \vec{h}^2} = \frac{1}{W}; \qquad w[C] = \frac{\vec{h}^2}{2 + c^2 * \vec{h}^2} \tag{10.121b}$$

The value at a local origin $\vec{\mathbf{0}}$ is determined. The domain of the difference $\vec{h}$ is determined by strict stability according to (10.107).

$$y\left[\vec{\mathbf{0}}\right] = \frac{1}{W} * y[L] + \frac{1}{W} * y[R] + w[C] * s\left[\vec{\mathbf{0}}\right] \tag{10.122}$$

An interpolation is determined by a number of polynomials joined by Dirichlet conditions.

$$y[0] = q[0] \tag{10.123a}$$

$$\sum^{1 \le i < n-1} \left\langle -\frac{1}{W} * y[i-1] + y[i] - \frac{1}{W} * y[i+1] = w[C] * s[i] \right\rangle \tag{10.123b}$$

$$y[n-1] = q[n-1] \tag{10.123c}$$

The equations of unknowns are normalized to the neighbours.

$$\sum^{1 \le i < n} \left\langle -y[i-1] + W * y[i] - y[i+1] = W * w[C] * s[i] = \vec{h}^2 * s[i] = q[i] \right\rangle \tag{10.124}$$

A symmetric uniform tridiagonal system of linear equations determines the unknowns.

$$\begin{bmatrix} W & -1 & & & & & & \\ -1 & W & -1 & & & & & \\ & -1 & W & -1 & & & & \\ & & & . \; . \; . & & & & \\ & & & & -1 & W & -1 & \\ & & & & & -1 & W & -1 \\ & & & & & & -1 & W \end{bmatrix} * \begin{bmatrix} y[1] \\ y[2] \\ y[3] \\ . \\ y[n-4] \\ y[n-3] \\ y[n-2] \end{bmatrix} = \begin{bmatrix} q[1] \\ q[2] \\ q[3] \\ . \\ q[n-4] \\ q[n-3] \\ q[n-2] \end{bmatrix} \tag{10.125}$$

A determinant according to (10.55) results the hyperbolic sine theorem.

$$W = 2 + c^2 * \vec{h}^2; \qquad d * \vec{h} = \operatorname{arcosh}\left(1 + \frac{1}{2} * c^2 * \vec{h}^2\right); \qquad 0 < \vec{h} \tag{10.126a}$$

$$D[j] = \sum^{0 \le i \le \frac{j-1}{2}} \left\{(-1)^i * \binom{j-1-i}{i} * W^{j-1-2*i}\right\} = \frac{\sinh\left(d * j * \vec{h}\right)}{\sinh\left(d * \vec{h}\right)} \tag{10.126b}$$

The determinant of a source matrix is defined according to (10.57). All signs cancel.

$$\begin{aligned}\det(Q[n-1][k]) = {} & D[n-1-k] * \sum^{0 \le i < k} \left\{(-1)^{k+i} * (-1)^{k-i} * q[i+1] * D[i+1]\right\} \\ & + D[k+1] * \sum^{k \le i < n-1} \left\{(-1)^{k+i} * (-1)^{i-k} * q[i+1] * D[n-1-i]\right\} \\ = {} & \frac{\sinh\left(d * (n-1-k) * \vec{h}\right)}{\sinh\left(d * \vec{h}\right)} * \sum^{0 \le i < k} \left\{q[i+1] * \frac{\sinh\left(d * (i+1) * \vec{h}\right)}{\sinh\left(d * \vec{h}\right)}\right\} \\ & + \frac{\sinh\left(d * (k+1) * \vec{h}\right)}{\sinh\left(d * \vec{h}\right)} * \sum^{k \le i < n-1} \left\{q[i+1] * \frac{\sinh\left(d * (n-1-i) * \vec{h}\right)}{\sinh\left(d * \vec{h}\right)}\right\}\end{aligned} \tag{10.127}$$

The determinant of the base matrix is defined.

$$D[n-1] = \frac{\sinh\left(d * (n-1) * \vec{h}\right)}{\sinh\left(d * \vec{h}\right)} \tag{10.128}$$

The solution is determined by Cramer's rule.

$$\sum^{0 \le k < n-2} \left\langle y[k+1] = \frac{\begin{aligned}&\sinh\left(d * (n-1-k) * \vec{h}\right) * \sum^{0 \le i < k} \left\{q[i+1] * \sinh\left(d * (i+1) * \vec{h}\right)\right\} \\ &+ \sinh\left(d * (k+1) * \vec{h}\right) * \sum^{k \le i < n-1} \left\{q[i+1] * \sinh\left(d * (n-1-i) * \vec{h}\right)\right\}\end{aligned}}{\sinh\left(d * (n-1) * \vec{h}\right)} \right\rangle \tag{10.129}$$

The interpolation results the same solution to Poisson's and Laplace's equation (10.94) as the sine interpolation if the distribution coefficient equals Zero.

10.9 Damped Sine Extrapolation

The damped sine approximation is a repeated uniform extrapolation of a damped sine operator.

The damped sine operator is determined by two Dirichlet conditions left to and at the origin and a damped sine condition at the origin.

$$\partial f[\langle 0\rangle]\left(-\vec{h}\right) = y[L] \tag{10.130}$$

$$\partial f[\langle 0\rangle]\left(\vec{\mathbf{0}}\right) = y[0] \tag{10.131}$$

$$\omega^2 * \partial f[\langle 0\rangle]\left(\vec{\mathbf{0}}\right) + 2 * \delta * \partial f[\langle 1\rangle]\left(\vec{\mathbf{0}}\right) + \partial f[\langle 2\rangle]\left(\vec{\mathbf{0}}\right) = 0 \tag{10.132}$$

The transposed polynomial of $\vec{h}$ is determined by a system of linear equations if the two points are not coincident.

$$\begin{bmatrix} 1 & 1 & \omega^2 \\ -\vec{h} & 0 & 2*\delta \\ \vec{h}^2 & 0 & 2 \end{bmatrix} * \begin{bmatrix} w[L] \\ w[0] \\ w[1] \end{bmatrix} = \begin{bmatrix} 1 \\ \vec{h} \\ \vec{h}^2 \end{bmatrix} \tag{10.133}$$

The base polynomials are determined.

$$w[L] = \frac{\delta * \vec{h} - 1}{\delta * \vec{h} + 1}; \qquad w[0] = \frac{2 - \omega^2 * \vec{h}^2}{\delta * \vec{h} + 1}; \qquad w[1] = \frac{\vec{h}^2}{\delta * \vec{h} + 1} \tag{10.134}$$

The value of the transposed polynomial of $\vec{h}$ is determined.

$$y\left[\vec{h}\right] = y[L] * w[L] + y[0] * w[0] \tag{10.135}$$

Suppose the solution is determined by two functions.

$$\text{if}\,(\omega > \delta)\text{then}\left(f(\vec{x}) = R * \mathsf{e}^{-\delta*\vec{x}} * \sin\left(\varphi + \vec{x} * \sqrt{\omega^2 - \delta^2}\right)\right) \tag{10.136}$$

$$\text{if}\,(\omega < \delta)\text{then}\left(g(\vec{x}) = R * \mathsf{e}^{-\delta*\vec{x}} * \sinh\left(\varphi + \vec{x} * \sqrt{\delta^2 - \omega^2}\right)\right) \tag{10.137}$$

Each function depends on two parameters ω and δ. The transposed polynomial only gives one equation in each case. A solution to this under-determined system is unknown.

$$\mathsf{e}^{-\delta*\vec{h}} * \sin\left(\varphi + \vec{h} * \sqrt{\omega^2 - \delta^2}\right) = \mathsf{e}^{\delta*\vec{h}} * \sin\left(\varphi - \vec{h} * \sqrt{\omega^2 - \delta^2}\right) * w[L] + \sin(\varphi) * w[0] \tag{10.138}$$

$$\mathsf{e}^{-\delta*\vec{h}} * \sinh\left(\varphi + \vec{h} * \sqrt{\omega^2 - \delta^2}\right) = \mathsf{e}^{\delta*\vec{h}} * \sinh\left(\varphi - \vec{h} * \sqrt{\omega^2 - \delta^2}\right) * w[L] + \sinh(\varphi) * w[0] \tag{10.139}$$

The extrapolation tends to the analytical damped sine for small differences $\vec{h}$.

$$\text{if}\,(\omega > \delta)\text{then}\left(\lim_{\vec{h}\to\vec{\mathbf{0}}}(\text{scw}(w[L]\,; w[0]\,; j; 1)) = f\left(j * \vec{h}\right) \Big/ f\left(-\vec{h}\right)\right) \tag{10.140}$$

$$\text{if}\,(\omega < \delta)\text{then}\left(\lim_{\vec{h}\to\vec{\mathbf{0}}}(\text{scw}(w[L]\,; w[0]\,; j; 1)) = g\left(j * \vec{h}\right) \Big/ g\left(-\vec{h}\right)\right) \tag{10.141}$$

It is assumed that the determinant of the base matrix defines the lower bound.

$$2 * \vec{h} * \left(1 - \delta * \vec{h}\right) \neq 0; \qquad \vec{h} \neq \vec{\mathbf{0}} \tag{10.142}$$

It is assumed that the upper bound is determined by the roots of the base polynomials. Therefore the extrapolation is stable if the base polynomials do not change signs.

$$\text{if}\left(\vec{h} \to \vec{\mathbf{0}}\right)\text{then}\left(\delta * \vec{h} - 1 < 0\right); \qquad \vec{h} < \frac{1}{\delta} \tag{10.143a}$$

$$\text{if}\left(\vec{h} \to \vec{\mathbf{0}}\right)\text{then}\left(2 - \omega^2 * \vec{h}^2 > 0\right); \qquad \vec{h} < \frac{\sqrt{2}}{\omega} \tag{10.143b}$$

10.10 Damped Sine Interpolation

The damped sine interpolation method approximates a damped sine on a uniformly discretized domain by a damped sine operator.

The simplest damped sine operator is determined by two Dirichlet conditions left and right to a base point or local origin and a damped sine condition of a source s at the base point.

$$\partial f[\langle 0\rangle]\left(-\vec{h}\right) = y[L] \tag{10.144}$$

$$\omega^2 * \partial f[\langle 0\rangle]\left(\vec{\mathbf{0}}\right) + 2 * \delta * \partial f[\langle 1\rangle]\left(\vec{\mathbf{0}}\right) + \partial f[\langle 2\rangle]\left(\vec{\mathbf{0}}\right) = s\left[\vec{\mathbf{0}}\right] \tag{10.145}$$

$$\partial f[\langle 0\rangle]\left(\vec{h}\right) = y[R] \tag{10.146}$$

The base polynomials of the operator are determined by a system of linear equations.

$$\begin{bmatrix} 1 & \omega^2 & 1 \\ -\vec{h} & 2 * \delta & \vec{h} \\ \vec{h}^2 & 2 & \vec{h}^2 \end{bmatrix} * \begin{bmatrix} w[L] \\ w[C] \\ w[R] \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{10.147a}$$

$$w[L] = \frac{1 - \delta * \vec{h}}{2 - \omega^2 * \vec{h}^2}; \qquad w[C] = -\frac{\vec{h}^2}{2 - \omega^2 * \vec{h}^2}; \qquad w[R] = \frac{1 + \delta * \vec{h}}{2 - \omega^2 * \vec{h}^2} \tag{10.147b}$$

The value at a local origin $\vec{\mathbf{0}}$ is determined. The domain of the difference $\vec{h}$ is determined by stability according to (10.143).

$$y\left[\vec{\mathbf{0}}\right] = w[L] * y[L] + w[R] * y[R] + w[C] * s\left[\vec{\mathbf{0}}\right] \tag{10.148}$$

An interpolation is determined by a number of polynomials joined by Dirichlet conditions.

$$y[0] = q[0] \tag{10.149a}$$

$$\sum^{1 \le i < n-1} \langle -w[L] * y[i-1] + y[i] - w[R] * y[i+1] = w[C] * s[i]\rangle \tag{10.149b}$$

$$y[n-1] = q[n-1] \tag{10.149c}$$

The equations of unknowns are modified.

$$W[I] = \frac{2 - \omega^2 * \vec{h}^2}{1 + \delta * \vec{h}}; \qquad W[L] = \frac{1 - \delta * \vec{h}}{1 + \delta * \vec{h}}; \qquad W[C] = -\frac{\vec{h}^2}{1 + \delta * \vec{h}} \tag{10.150a}$$

$$\sum^{1 \le i < n} \langle -W[L] * y[i-1] + W[I] * y[i] - y[i+1] = W[C] * s[i] = q[i]\rangle \tag{10.150b}$$

A uniform tridiagonal system of linear equations determines the unknowns.

$$\begin{bmatrix} W[I] & -1 & & & & & & \\ W[L] & W[I] & -1 & & & & & \\ & W[L] & W[I] & -1 & & & & \\ & & & & . \quad . \quad . & & & \\ & & & & W[L] & W[I] & -1 & \\ & & & & & W[L] & W[I] & -1 \\ & & & & & & W[L] & W[I] \end{bmatrix} * \begin{bmatrix} y[1] \\ y[2] \\ y[3] \\ . \\ y[n-4] \\ y[n-3] \\ y[n-2] \end{bmatrix} = \begin{bmatrix} q[1] \\ q[2] \\ q[3] \\ . \\ q[n-4] \\ q[n-3] \\ q[n-2] \end{bmatrix} \tag{10.151}$$

A determinant is defined according to (10.55).

$$D[j] = \sum^{0 \le i \le \frac{j-1}{2}} \left\{ \binom{j-1-i}{i} * W[L]^i * W[I]^{j-1-2*i} \right\}; \qquad j \ge 1 \tag{10.152}$$

Suppose the solution is defined in terms of two functions.

$$\text{if}\,(\omega > \delta)\text{then}\left(f(\vec{x}) = R * \mathsf{e}^{-\delta*\vec{x}} * \sin\left(\varphi + \vec{x} * \sqrt{\omega^2 - \delta^2}\right)\right) \tag{10.153a}$$

$$\text{if}\,(\omega < \delta)\text{then}\left(g(\vec{x}) = R * \mathsf{e}^{-\delta*\vec{x}} * \sinh\left(\varphi + \vec{x} * \sqrt{\delta^2 - \omega^2}\right)\right) \tag{10.153b}$$

The solution approximates a composed damped sine according to (10.57) and (10.58).

$$\text{if}\,(\omega > \delta)\text{then}\left(\lim_{\vec{h}\to\vec{\mathbf{0}}}(D[j]) = f\left(j * \vec{h}\right) \Big/ f\left(-\vec{h}\right)\right) \tag{10.154a}$$

$$\text{if}\,(\omega < \delta)\text{then}\left(\lim_{\vec{h}\to\vec{\mathbf{0}}}(D[j]) = g\left(j * \vec{h}\right) \Big/ g\left(-\vec{h}\right)\right) \tag{10.154b}$$

Listing 10.1: sine theorem in C with gmp [4]

```
#include <assert.h>
#include <math.h>
#include <stdio.h>
#include <gmp.h>

void gbinom(mpf_t r, unsigned const a, unsigned const b)
{
  unsigned i;
  mpf_set_ui(r, 1);
  for(i = 1; i <= b; ++i)
  {
    mpf_mul_ui(r, r, a-i+1);
    mpf_div_ui(r, r, i);
  }
}

int main(void)
{
  double const c = 3.,h = .2, w0 = 2.-h*h*c*c;
  double const d = acos(w0/2.)/h, r = 1./sin(d*h);
  unsigned i, j;
  double s;
  mpf_t e, b, t;
  FILE * f = fopen("gsine.dat", "w");
  assert(f);
  mpf_set_default_prec(1024);
  mpf_init(b); mpf_init(e); mpf_init(t);
  for(j = 1; j < 500; ++j)
  {
    mpf_set_ui(e, 0);
    for(i = 0; 2*i <= j-1; ++i)
    {
      gbinom(b, j-i-1, i);
      if(i%2) { mpf_neg(b, b); }
      mpf_set_d(t, w0);
      mpf_pow_ui(t, t, j-1-2*i);
      mpf_mul(b, b, t);
      mpf_add(e, e, b);
    }
    s = sin(d*(j)*h)*r;
    fprintf(f, "%f %f %f %.24f\n", j*h, mpf_get_d(e), s, mpf_get_d(e)-s);
  }
  fclose(f);
  mpf_clear(b); mpf_clear(e); mpf_clear(t);
  printf("%f*sin(%f*x)\n", r, d);
  return 0;
}
```

Listing 10.2: sine interpolation in C with gsl [5]

```
#include <assert.h>
#include <math.h>
#include <stdio.h>
#include <gsl/gsl_linalg.h>

int main(void)
{
  unsigned const n = 20;
  double const L = 1., h = L / n;
  double const yl = -2., yr = 5.;
  double const c = 4.;
  double const W = 2. - c*c*h*h;
  double const d = 1./h*acos(1. - .5*c*c*h*h);
  unsigned i;
  int s;
  double ya;
  FILE * f;
  gsl_matrix * G = gsl_matrix_calloc(n+1, n+1);
  gsl_vector * y = gsl_vector_alloc(n+1);
  gsl_vector * q = gsl_vector_calloc(n+1);
  gsl_permutation * p = gsl_permutation_alloc(n+1);
  // left boundary
  gsl_matrix_set(G, 0, 0, 1.); gsl_vector_set(q, 0, yl);
  // inside
  for(i = 1; i < n; ++i)
  {
    gsl_matrix_set(G, i, i-1, 1.);
    gsl_matrix_set(G, i, i, -W);
    gsl_matrix_set(G, i, i+1, 1.);
  }
  // right boundary
  gsl_matrix_set(G, n, n, 1.); gsl_vector_set(q, n, yr);
  //
  gsl_linalg_LU_decomp(G, p, &s);
  gsl_linalg_LU_solve(G, p, q, y);
  //
  f = fopen("sine.interpolation.dat", "w");
  for(i = 0; f && i <= n; ++i)
  {
    ya = (sin(d*(n-i)*h)*yl + sin(d*i*h)*yr)/sin(d*n*h);
    fprintf(f, "%f %f %f\n", i*h, gsl_vector_get(y, i), ya);
  }
  fclose(f);
  //
  gsl_matrix_free(G); gsl_vector_free(y); gsl_vector_free(q);
  gsl_permutation_free(p);
  //
  return 0;
}
```

Chapter 11

Logarithm

The natural logarithm is the unknown integral of a hyperbola.

$$y = \log[\mathsf{e}](\vec{x})\,; \qquad \partial\left(1; \log[\mathsf{e}](\vec{x})\,; \vec{u}\right) = \frac{1}{\vec{u}}; \qquad \operatorname{size}(\vec{x}) = 1 \tag{11.1}$$

Logarithm and exponential funcition are inverse.

$$\exp((\vec{x})) = \mathsf{e}^{\vec{x}}; \qquad \log[\mathsf{e}]\left(\mathsf{e}^{\vec{x}}\right) = \vec{x} \tag{11.2}$$

Logarithms of another base than e are multiples of the natural logarithm.

$$b^{y} = \vec{x}; \qquad y = \log[b](\vec{x}) = \frac{\log[\mathsf{e}](\vec{x})}{\log[\mathsf{e}](b)}; \qquad \partial\left(1; \log[b](\vec{x})\,; \vec{u}\right) = \frac{1}{\vec{x} * \log[\mathsf{e}](b)} \tag{11.3}$$

Logarithms obey certain laws, [2, p260,logarithm].

$$\log[b](\vec{x}^{c}) = c * \log[b](\vec{x})\,; \qquad \log[b](c * \vec{x}) = \log[b](c) + \log[b](\vec{x}) \tag{11.4}$$

The value of a logarithm may be computed explicitly and implicitly. A number of explicit approximations to the logarithm form series, see [3, p633].

11.1 Extrapolation by the Base Point

The terms of a polynomial are determined.

$$f(\vec{x}) = \sum^{0 \le i < N} \left\{a[i] * \vec{x}^{i}\right\}; \qquad \partial f[j](\vec{u}) = \partial\left(j; f(\vec{x})\,; \vec{u}\right); \qquad \operatorname{size}(\vec{x}) = 1 \tag{11.5}$$

A base point and the derivatives of the logaritm are defined.

$$\partial f[0]\left(\vec{U}\right) = Y; \qquad \partial f[j]\left(\vec{U}\right) = (-1)^{j-1} * \frac{(j-1)!}{\vec{U}^{j}}; \qquad j > 0 \tag{11.6}$$

The transposed polynomial is determined.

$$f(\vec{x}) = \sum^{0 \le j < N} \left\{ w[j](\vec{x}) * \partial f[j]\left(\vec{U}\right) \right\} \tag{11.7}$$

The weights or base polynomials are determined by a system of linear equations.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \vec{U} & 1 & 0 & 0 & \cdots \\ \vec{U}^2 & 2 * \vec{U} & 2 & 0 & \cdots \\ \vec{U}^3 & 3 * \vec{U}^2 & 6 * \vec{U} & 6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} * \begin{bmatrix} w[0](\vec{x}) \\ w[1](\vec{x}) \\ w[2](\vec{x}) \\ w[3](\vec{x}) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ \vec{x} \\ \vec{x}^2 \\ \vec{x}^3 \\ \vdots \end{bmatrix} \tag{11.8}$$

The base matrix is triangular such that the solution is available explicitly.

$$w[0](\vec{x}) = 1 \tag{11.9}$$

$$w[1](\vec{x}) = \vec{x} - \vec{U} * w[0](\vec{x}) \tag{11.10}$$

$$w[2](\vec{x}) = \frac{1}{2} * \left(\vec{x}^2 - \vec{U}^2 * w[0](\vec{x}) - 2 * \vec{U} * w[1](\vec{x})\right) \tag{11.11}$$

$$w[3](\vec{x}) = \frac{1}{6} * \left(\vec{x}^3 - \vec{U}^3 * w[0](\vec{x}) - 3 * \vec{U}^2 * w[1](\vec{x}) - 6 * \vec{U} * w[2](\vec{x})\right) \tag{11.12}$$

$$w[m](\vec{x}) = \frac{1}{m!} * \left(\vec{x}^m - \sum^{0 \le k < m} \left\{ (m\text{¡}k) * \vec{U}^{m-k} * w[k](\vec{x}) \right\} \right) \tag{11.13}$$

The weights are substituted.

$$w[0](\vec{x}) = 1 \tag{11.14}$$

$$w[1](\vec{x}) = \vec{x} - \vec{U} \tag{11.15}$$

$$w[2](\vec{x}) = \frac{1}{2!} * \left(\vec{x}^2 - \vec{U}^2 - 2 * \vec{U} * \left(\vec{x} - \vec{U}\right)\right) = \frac{1}{2!} * \left(\vec{x} - \vec{U}\right)^2 \tag{11.16}$$

$$w[3](\vec{x}) = \frac{1}{3!} * \left(\vec{x}^3 - \vec{U}^3 - 3 * \vec{U}^2 * \left(\vec{x} - \vec{U}\right) - \frac{(3\text{¡}2)}{2!} * \vec{U} * \left(\vec{x} - \vec{U}\right)^2\right) = \frac{1}{3!} * \left(\vec{x} - \vec{U}\right)^3 \tag{11.17}$$

$$w[m](\vec{x}) = \frac{1}{m!} * \left(\vec{x}^m - \sum^{0 \le k < m} \left\{ \binom{m}{k} * \vec{U}^{m-k} * \left(\vec{x} - \vec{U}\right)^k \right\} \right) = \frac{1}{m!} * \left(\vec{x} - \vec{U}\right)^m \tag{11.18}$$

The solution of the weights is substituted into the transposed polynomial.

$$f(\vec{x}) = \sum^{0 \le j < N} \left\{ \frac{1}{j!} * \left(\vec{x} - \vec{U}\right)^j * \partial f[j]\left(\vec{U}\right) \right\} \tag{11.19}$$

The terms of the derivatives are substituted into the transposed polynomial.

$$f(\vec{x}) = Y + \sum^{1 \le j < N} \left\{ \frac{1}{j!} * \left(\vec{x} - \vec{U}\right)^j * (-1)^{j-1} * \frac{(j-1)!}{\vec{U}^j} \right\} \tag{11.20}$$

The transposed polynomial results a series.

$$f(\vec{x}) = Y + \sum^{1 \le j < N} \left\{ (-1)^{j-1} * \frac{\left(\vec{x} - \vec{U}\right)^j}{j * \vec{U}^j} \right\} \tag{11.21}$$

The series is checked against strict stability (2.48a).

$$\text{abs}\left(\frac{\left(\vec{x}-\vec{U}\right)^{j}}{j*\vec{U}^{j}}\right) > 2*\text{abs}\left(\frac{\left(\vec{x}-\vec{U}\right)^{j+1}}{(j+1)*\vec{U}^{j+1}}\right) \tag{11.22a}$$

$$\text{abs}\left(\frac{j+1}{j}*\vec{U}\right) > 2*\text{abs}\left(\vec{x}-\vec{U}\right) \tag{11.22b}$$

$$\text{abs}\left(\vec{U}\right) > 2*\text{abs}\left(\vec{x}-\vec{U}\right) \tag{11.22c}$$

The series is strictly stable and thus mostly determined by the first terms if the distance of base point and interpolation point is less than half the position of the base point.

$$\log[\mathsf{e}]\left[\vec{U}\right](\vec{x}) \approx Y + \sum^{1\leq j<N}\left\{(-1)^{j-1}*\frac{\left(\vec{x}-\vec{U}\right)^{j}}{j*\vec{U}^{j}}\right\}; \quad \text{abs}\left(\vec{U}\right) > 2*\text{abs}\left(\vec{x}-\vec{U}\right) \tag{11.23}$$

Base points may be determined by the exponential function.

$$\mathsf{e}^2 > 2*\text{abs}\left(10-\mathsf{e}^2\right); \quad \log[\mathsf{e}]\left[\mathsf{e}^2\right]\left(\vec{10}\right) \approx 2 + \sum^{1\leq j<3}\left\{(-1)^{j-1}*\frac{\left(\vec{10}-\vec{\mathsf{e}}^2\right)^{j}}{j*\vec{\mathsf{e}}^{2*j}}\right\} \approx 2.305630 \tag{11.24}$$

$$\mathsf{e}^4 > 2*\text{abs}\left(50-\mathsf{e}^4\right); \quad \log[\mathsf{e}]\left[\mathsf{e}^4\right]\left(\vec{50}\right) \approx 4 + \sum^{1\leq j<3}\left\{(-1)^{j-1}*\frac{\left(\vec{50}-\vec{\mathsf{e}}^4\right)^{j}}{j*\vec{\mathsf{e}}^{4*j}}\right\} \approx 3.912036 \tag{11.25}$$

11.1.1 Application of Logarithmic Identity

Different series may be combined by the logarithmic identities. A special case is discussed of combining two series of opposite differences.

$$f\left((1-s)*\vec{U}\right) = Y + \sum^{1\leq j<N}\left\{(-1)^{j-1}*\frac{\left((1-s)*\vec{U}-\vec{U}\right)^{j}}{j*\vec{U}^{j}}\right\} \tag{11.26}$$

$$f\left((1+s)*\vec{U}\right) = Y + \sum^{1\leq j<N}\left\{(-1)^{j-1}*\frac{\left((1+s)*\vec{U}-\vec{U}\right)^{j}}{j*\vec{U}^{j}}\right\} \tag{11.27}$$

The base position cancels out of the terms.

$$f\left((1-s)*\vec{U}\right) = Y + \sum^{1\leq j<N}\left\{(-1)^{j-1}*\frac{(-s)^{j}}{j}\right\} = Y - \sum^{1\leq j<N}\left\{\frac{s^{j}}{j}\right\} \tag{11.28}$$

$$f\left((1+s)*\vec{U}\right) = Y + \sum^{1\leq j<N}\left\{(-1)^{j-1}*\frac{s^{j}}{j}\right\} = Y - \sum^{1\leq j<N}\left\{\frac{(-s)^{j}}{j}\right\} \tag{11.29}$$

The difference of the series is determined.

$$f\Big((1+s)*\vec{U}\Big) - f\Big((1-s)*\vec{U}\Big) = -\sum^{1\le j<N}\left\{\frac{(-s)^j}{j} - \frac{s^j}{j}\right\}$$
$$= 2*\sum^{0\le j<N/2}\left\{\frac{s^{2*j+1}}{2*j+1}\right\} = g(s) \quad (11.30)$$

The difference of the logarithm is determined.

$$\log[\mathsf{e}]\Big((1+s)*\vec{U}\Big) - \log[\mathsf{e}]\Big((1-s)*\vec{U}\Big) = \log[\mathsf{e}]\left(\frac{(1+s)*\vec{U}}{(1-s)*\vec{U}}\right) = \log[\mathsf{e}]\left(\frac{1+s}{1-s}\right) \quad (11.31)$$

The argument of the logarithm is transformed.

$$\vec{x} = \frac{1+s}{1-s} \quad \longleftrightarrow \quad s = \frac{\vec{x}-1}{\vec{x}+1} \quad (11.32)$$

The transformation is substituted into the approximation.

$$g\left(\frac{\vec{x}-1}{\vec{x}+1}\right) = 2*\sum^{0\le j<N/2}\left\{\frac{1}{2*j+1}*\left(\frac{\vec{x}-1}{\vec{x}+1}\right)^{2*j+1}\right\} = h(\vec{x}) \quad (11.33)$$

The series is checked against strict stability (2.48a) within the domain of the logarithm $\vec{x} > 0$.

$$\text{abs}\left(\frac{1}{2*j+1}*\left(\frac{\vec{x}-1}{\vec{x}+1}\right)^{2*j+1}\right) > \text{abs}\left(\frac{2}{2*(j+1)+1}*\left(\frac{\vec{x}-1}{\vec{x}+1}\right)^{2*(j+1)+1}\right) \quad (11.34a)$$
$$1 > \frac{4*j+1}{2*j+3}*\left(\frac{\vec{x}-1}{\vec{x}+1}\right)^2 < 2*\left(\frac{\vec{x}-1}{\vec{x}+1}\right)^2 \quad (11.34b)$$
$$3-\sqrt{8} < \vec{x} < 3+\sqrt{8} \quad (11.34c)$$

The series is strictly stable only in a small domain.

$$\log[\mathsf{e}](\vec{x}) \approx 2*\sum^{0\le j<N/2}\left\{\frac{1}{2*j+1}*\left(\frac{\vec{x}-1}{\vec{x}+1}\right)^{2*j+1}\right\}; \qquad 3-\sqrt{8} < \vec{x} < 3+\sqrt{8} \quad (11.35)$$

The series converges slowly outside the stable domain compared to (11.24) and (11.25).

$$\log[\mathsf{e}]\left(\vec{10}\right) \approx 2*\sum^{0\le j<21/2}\left\{\frac{1}{2*j+1}*\left(\frac{\vec{10}-1}{\vec{10}+1}\right)^{2*j+1}\right\} \approx 2.300312 \quad (11.36)$$
$$\log[\mathsf{e}]\left(\vec{50}\right) \approx 2*\sum^{0\le j<101/2}\left\{\frac{1}{2*j+1}*\left(\frac{\vec{50}-1}{\vec{50}+1}\right)^{2*j+1}\right\} \approx 3.908560 \quad (11.37)$$

The logarithmic identity may be applied repeatedly.

11.2 Extrapolation by the Extrapolation Point

The terms of a polynomial are determined.

$$f(\vec{x}) = \sum^{0\leq i<N} \left\{a[i] * \vec{x}^i\right\}; \qquad \partial f[j](\vec{u}) = \partial\left(j; f(\vec{x}); \vec{u}\right); \qquad \text{size}(\vec{x}) = 1 \tag{11.38}$$

A base point and the derivatives at the extrapolation point are determined.

$$\partial f[0]\left(\vec{U}\right) = Y; \qquad \partial f[j](\vec{u}) = (-1)^{j-1} * \frac{(j-1)!}{\vec{u}^j}; \qquad j > 0 \tag{11.39}$$

The transposed polynomial is determined.

$$f(\vec{x}) = w[0](\vec{x}) * \partial f[0]\left(\vec{U}\right) + \sum^{1\leq j<N} \{w[j](\vec{x}) * \partial f[j](\vec{u})\} \tag{11.40}$$

The base polynomials are determined by a triangular system of equations.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ \vec{U} & 1 & 0 & 0 & \dots \\ \vec{U}^2 & 2*\vec{u} & 2 & 0 & \dots \\ \vec{U}^3 & 3*\vec{u}^2 & 6*\vec{u} & 6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} * \begin{bmatrix} w[0](\vec{x}) \\ w[1](\vec{x}) \\ w[2](\vec{x}) \\ w[3](\vec{x}) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ \vec{x} \\ \vec{x}^2 \\ \vec{x}^3 \\ \vdots \end{bmatrix} \tag{11.41}$$

The polynomial is not determined by its Taylor series. The polynomial is determined by differential conditions according to the logarithm. Therefore it is possible to equate $\vec{u}$ and $\vec{x}$ which simplifies the solution.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ \vec{U} & 1 & 0 & 0 & \dots \\ \vec{U}^2 & 2*\vec{x} & 2 & 0 & \dots \\ \vec{U}^3 & 3*\vec{x}^2 & 6*\vec{x} & 6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} * \begin{bmatrix} w[0](\vec{x}) \\ w[1](\vec{x}) \\ w[2](\vec{x}) \\ w[3](\vec{x}) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ \vec{x} \\ \vec{x}^2 \\ \vec{x}^3 \\ \vdots \end{bmatrix} \tag{11.42}$$

The solution is determined explicitly.

$$w[0](\vec{x}) = 1 \tag{11.43}$$

$$w[1](\vec{x}) = \vec{x} - \vec{U} * w[0](\vec{x}) \tag{11.44}$$

$$w[2](\vec{x}) = \frac{1}{2} * \left(\vec{x}^2 - \vec{U}^2 * w[0](\vec{x}) - 2 * \vec{x} * w[1](\vec{x})\right) \tag{11.45}$$

$$w[3](\vec{x}) = \frac{1}{6} * \left(\vec{x}^3 - \vec{U}^3 * w[0](\vec{x}) - 3 * \vec{x}^2 * w[1](\vec{x}) - 6 * \vec{x} * w[2](\vec{x})\right) \tag{11.46}$$

$$w[m](\vec{x}) = \frac{1}{m!} * \left(\vec{x}^m - \vec{U}^m * w[0](\vec{x}) - \sum^{1\leq k<m} \left\{ (m¡k) * \vec{x}^{m-k} * w[k](\vec{x})\right\}\right) \tag{11.47}$$

The base polynomials are substituted into the transposed polynomial.

$$f(\vec{x}) = \partial f[0]\left(\vec{U}\right) + \sum^{1\leq j<N} \left\{\frac{(-1)^{j+1}}{j!} * \left(\vec{x} - \vec{U}\right)^j * \partial f[j](\vec{u})\right\} \tag{11.48}$$

The terms of the derivatives of the extrapolation point are substituted.

$$f(\vec{x}) = Y + \sum^{1\leq j<N} \left\{ \frac{(-1)^{j+1}}{j!} * \left(\vec{x} - \vec{U}\right)^j * (-1)^{j-1} * \frac{(j-1)!}{\vec{x}^j} \right\} \tag{11.49}$$

The transposed polynomial results a series.

$$f(\vec{x}) = Y + \sum^{1\leq j<N} \left\{ \frac{\left(\vec{x} - \vec{U}\right)^j}{j * \vec{x}^j} \right\} \tag{11.50}$$

The series is checked against strict stability (2.48a) within the domain of the logarithm $\vec{x} > 0$.

$$\text{abs}\left(\frac{\left(\vec{x} - \vec{U}\right)^j}{j * \vec{x}^j} \right) > \text{abs}\left(\frac{\left(\vec{x} - \vec{U}\right)^{j+1}}{(j+1) * \vec{x}^{j+1}} \right) \tag{11.51a}$$

$$\vec{x} > 2 * \frac{j}{j+1} * \left(\vec{x} - \vec{U}\right) < 2 * \left(\vec{x} - \vec{U}\right) \tag{11.51b}$$

$$\vec{x} < 2 * \vec{U} \tag{11.51c}$$

The series is strictly stable and thus mostly determined by the first terms if the extrapolation position is less than double the base position.

$$\log[\mathsf{e}]\left[\vec{U}\right](\vec{x}) \approx Y + \sum^{1\leq j<N} \left\{ \frac{\left(\vec{x} - \vec{U}\right)^j}{j * \vec{x}^j} \right\}; \qquad \vec{\mathbf{0}} < \vec{x} < 2 * \vec{U} \tag{11.52}$$

A series determined by the application of logarithmic identities to this series is unknown.

11.3 Extrapolation by Both Points

The terms of a polynomial are determined.

$$f(\vec{x}) = \sum^{0\leq i<N} \left\{ a[i] * \vec{x}^i \right\}; \qquad \partial f[j](\vec{u}) = \partial\left(j; f(\vec{x}); \vec{u}\right); \qquad \text{size}\left(\vec{x}\right) = 1 \tag{11.53}$$

A base point and the derivatives of the logarithm are determined at both the base point and the extrapolation point.

$$\partial f[0]\left(\vec{U}\right) = Y \tag{11.54}$$

$$\partial f[j]\left(\vec{U}\right) = (-1)^{j-1} * \frac{(j-1)!}{\vec{U}^j}; \qquad \partial f[j](\vec{x}) = (-1)^{j-1} * \frac{(j-1)!}{\vec{x}^j}; \qquad j > 0 \tag{11.55}$$

A series is unknown. A number of approximations to the conditions is discussed.

$$f[m](\vec{x}) = w[0](\vec{x}) * \partial f[0]\left(\vec{U}\right) + \sum^{1\leq j\leq m} \left\{ \begin{array}{l} w[m][2*j-1](\vec{x}) * \partial f[j]\left(\vec{U}\right) \\ + w[m][2*j](\vec{x}) * \partial f[j](\vec{x}) \end{array} \right\} \tag{11.56}$$

The approximation of degree One is determined.

$$\begin{bmatrix} 1 & 0 & 0 \\ \vec{U} & 1 & 1 \\ \vec{U}^2 & 2*\vec{U} & 2*\vec{x} \end{bmatrix} * \begin{bmatrix} w[1][0](\vec{x}) \\ w[1][1](\vec{x}) \\ w[1][2](\vec{x}) \end{bmatrix} = \begin{bmatrix} 1 \\ \vec{x} \\ \vec{x}^2 \end{bmatrix}; \qquad \begin{matrix} w[1][0](\vec{x}) = 1 \\ w[1][1](\vec{x}) = (\vec{x}-\vec{U})/2 \\ w[1][2](\vec{x}) = (\vec{x}-\vec{U})/2 \end{matrix} \tag{11.57}$$

The approximation of degree One results a rational function.

$$f[1](\vec{x}) = Y + \frac{\vec{x}-\vec{U}}{2} * \left(\frac{1}{\vec{x}} + \frac{1}{\vec{U}}\right) = Y + \frac{\vec{x}-\vec{U}}{\vec{U}} - \frac{\left(\vec{x}-\vec{U}\right)^2}{2*\vec{U}*\vec{x}} \tag{11.58}$$

The approximation of degree Two is determined.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \vec{U} & 1 & 1 & 0 & 0 \\ \vec{U}^2 & 2*\vec{U} & 2*\vec{x} & 2 & 2 \\ \vec{U}^3 & 3*\vec{U}^2 & 3*\vec{x}^2 & 6*\vec{U} & 6*\vec{x} \\ \vec{U}^4 & 4*\vec{U}^3 & 4*\vec{x}^3 & 12*\vec{U}^2 & 12*\vec{x}^2 \end{bmatrix} * \begin{bmatrix} w[2][0](\vec{x}) \\ w[2][1](\vec{x}) \\ w[2][2](\vec{x}) \\ w[2][3](\vec{x}) \\ w[2][4](\vec{x}) \end{bmatrix} = \begin{bmatrix} 1 \\ \vec{x} \\ \vec{x}^2 \\ \vec{x}^3 \\ \vec{x}^4 \end{bmatrix} \tag{11.59}$$

The base polynomials of the approximation of second degree are determined.

$$w[2][0](\vec{x}) = 1 \tag{11.60}$$

$$w[2][1](\vec{x}) = (\vec{x}-\vec{U})/2; \qquad w[2][2](\vec{x}) = (\vec{x}-\vec{U})/2 \tag{11.61}$$

$$w[2][3](\vec{x}) = (\vec{x}-\vec{U})^2/12; \qquad w[2][4](\vec{x}) = -(\vec{x}-\vec{U})^2/12 \tag{11.62}$$

The second degree approximation contains the first degree approximation.

$$f[2](\vec{x}) = Y + \frac{\vec{x}-\vec{U}}{2} * \left(\frac{1}{\vec{x}} + \frac{1}{\vec{U}}\right) + \frac{\left(\vec{x}-\vec{U}\right)^2}{12} * \left(\frac{1}{\vec{x}^2} - \frac{1}{\vec{U}^2}\right) \tag{11.63}$$

$$f[2](\vec{x}) = f[1](\vec{x}) - \frac{\left(\vec{x}-\vec{U}\right)^3}{6*\vec{U}^2*\vec{x}} + \frac{\left(\vec{x}-\vec{U}\right)^4}{12*\vec{U}^2*\vec{x}^2} \tag{11.64}$$

The approximation of degree Three is determined.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vec{U} & 1 & 1 & 0 & 0 & 0 & 0 \\ \vec{U}^2 & 2*\vec{U} & 2*\vec{x} & 2 & 2 & 0 & 0 \\ \vec{U}^3 & 3*\vec{U}^2 & 3*\vec{x}^2 & 6*\vec{U} & 6*\vec{x} & 6 & 6 \\ \vec{U}^4 & 4*\vec{U}^3 & 4*\vec{x}^3 & 12*\vec{U}^2 & 12*\vec{x}^2 & 24*\vec{U} & 24*\vec{x} \\ \vec{U}^5 & 5*\vec{U}^4 & 5*\vec{x}^4 & 20*\vec{U}^3 & 20*\vec{x}^3 & 60*\vec{U}^2 & 60*\vec{x}^2 \\ \vec{U}^6 & 6*\vec{U}^5 & 6*\vec{x}^5 & 30*\vec{U}^4 & 30*\vec{x}^4 & 120*\vec{U}^3 & 120*\vec{x}^3 \end{bmatrix} * \begin{bmatrix} w[3][0](\vec{x}) \\ w[3][1](\vec{x}) \\ w[3][2](\vec{x}) \\ w[3][3](\vec{x}) \\ w[3][4](\vec{x}) \\ w[3][5](\vec{x}) \\ w[3][6](\vec{x}) \end{bmatrix} = \begin{bmatrix} 1 \\ \vec{x} \\ \vec{x}^2 \\ \vec{x}^3 \\ \vec{x}^4 \\ \vec{x}^5 \\ \vec{x}^6 \end{bmatrix} \tag{11.65}$$

The base polynomials of the approximation of third degree are determined.

$$w[3][0](\vec{x}) = 1 \tag{11.66}$$

$$w[3][1](\vec{x}) = (\vec{x}-\vec{U})/2; \qquad w[3][2](\vec{x}) = (\vec{x}-\vec{U})/2 \tag{11.67}$$

$$w[3][3](\vec{x}) = (\vec{x}-\vec{U})^2/10; \qquad w[3][4](\vec{x}) = -(\vec{x}-\vec{U})^2/10 \tag{11.68}$$

$$w[3][5](\vec{x}) = (\vec{x}-\vec{U})^3/120; \qquad w[3][6](\vec{x}) = (\vec{x}-\vec{U})^3/120 \tag{11.69}$$

The third degree approximation contains the second degree approximation.

$$f[3](\vec{x}) = Y + \frac{\vec{x}-\vec{U}}{2} * \left(\frac{1}{\vec{x}} + \frac{1}{\vec{U}}\right) + \frac{\left(\vec{x}-\vec{U}\right)^2}{10} * \left(\frac{1}{\vec{x}^2} - \frac{1}{\vec{U}^2}\right) + \frac{\left(\vec{x}-\vec{U}\right)^3}{60} * \left(\frac{1}{\vec{x}^3} + \frac{1}{\vec{U}^3}\right) \tag{11.70}$$

$$f[3](\vec{x}) = f[2](\vec{x}) + \frac{\left(\vec{x}-\vec{U}\right)^5}{30 * \vec{U}^3 * \vec{x}^2} - \frac{\left(\vec{x}-\vec{U}\right)^6}{60 * \vec{U}^4 * \vec{x}^4} \tag{11.71}$$

The fourth degree approximation does not contain the third degree approximation such that a series is unknown. A general approximation is determined.

$$f[m](\vec{x}) = Y + \sum^{1 \le j \le m} \left\{ b[m][j] * \left(\vec{x}-\vec{U}\right)^j * \left(\frac{1}{\vec{x}^j} + \frac{(-1)^{j+1}}{\vec{U}^j}\right) \right\} \tag{11.72}$$

The approximations up to degree Six are determined by coefficients.

$$b[0] = (1)\,; \qquad b[1] = \left(1; \frac{1}{2}\right) \tag{11.73}$$

$$b[2] = \left(1; \frac{1}{2}; \frac{1}{12}\right); \qquad b[3] = \left(1; \frac{1}{2}; \frac{1}{10}; \frac{2}{120}\right) \tag{11.74}$$

$$b[4] = \left(1; \frac{1}{2}; \frac{1}{28}; \frac{2}{84}; \frac{6}{1680}\right); \qquad b[5] = \left(1; \frac{1}{2}; \frac{1}{9}; \frac{2}{72}; \frac{6}{1008}; \frac{24}{30240}\right) \tag{11.75}$$

$$b[6] = \left(1; \frac{1}{2}; \frac{1}{44}; \frac{2}{66}; \frac{6}{792}; \frac{24}{15840}; \frac{120}{665280}\right) \tag{11.76}$$

$$\tag{11.77}$$

Stability cannot be checked generally. The third degree approximation is evaluated for two points.

$$\mathsf{e}^2 \approx 10; \quad \log[\mathsf{e}]\left(\vec{10}\right) \approx 2 + \sum^{1 \le j < 3} \left\{ b[3][j] * \left(\vec{10} - \vec{\mathsf{e}}^2\right)^j * \left(\frac{1}{\vec{10}^j} + \frac{(-1)^{j+1}}{\vec{\mathsf{e}}^{2*j}}\right) \right\} = 2.302587 \tag{11.78}$$

$$\mathsf{e}^4 \approx 50; \quad \log[\mathsf{e}]\left(\vec{50}\right) \approx 4 + \sum^{1 \le j < 3} \left\{ b[3][j] * \left(\vec{50} - \vec{\mathsf{e}}^2\right)^j * \left(\frac{1}{\vec{50}^j} + \frac{(-1)^{j+1}}{\vec{\mathsf{e}}^{2*j}}\right) \right\} = 3.912023 \tag{11.79}$$

11.4 Interpolation

A approximation by interpolation with a simple solution is unknown. Two cases are discussed. The terms of a polynomial are determined.

$$f(\vec{x}) = \sum^{0 \le i < N} \left\{ a[i] * \vec{x}^i \right\}; \qquad \partial f[j](\vec{u}) = \partial\left(j; f(\vec{x}); \vec{u}\right); \qquad \text{size}\left(\vec{x}\right) = 1 \tag{11.80}$$

11.4.1 Derivatives at the Interpolation Point

Two base points and a number of derivatives at the interpolation point are determined.

$$\partial f[0]\left(\vec{U}\right) = Y; \tag{11.81}$$

$$\partial f[j](\vec{x}) = (-1)^{j-1} * \frac{(j-1)!}{\vec{x}^j}; \qquad \vec{U} < \vec{x}; \qquad j > 0 \tag{11.82}$$

$$\partial f[0]\left(\vec{V}\right) = Z; \qquad \vec{x} < \vec{V} \tag{11.83}$$

The transposed polynomial is determined.

$$f(\vec{x}) = w[0]\left(\vec{U}\right) * Y + w[1]\left(\vec{V}\right) * Z + \sum^{2 \le j < N} \left\{ w[j](\vec{x}) * \partial f[j]\left(\vec{U}\right) \right\} \tag{11.84}$$

The base polynomials are determined by a system of linear equations.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \dots \\ \vec{U} & \vec{V} & 1 & 0 & 0 & \dots \\ \vec{U}^2 & \vec{V}^2 & 2 * \vec{x} & 2 & 0 & \dots \\ \vec{U}^3 & \vec{V}^3 & 3 * \vec{x}^2 & 6 * \vec{x} & 0 & \dots \\ \vec{U}^4 & \vec{V}^4 & 4 * \vec{x}^3 & 12 * \vec{x}^2 & 24 * \vec{x} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} * \begin{bmatrix} w[0](\vec{x}) \\ w[1](\vec{x}) \\ w[2](\vec{x}) \\ w[3](\vec{x}) \\ w[4](\vec{x}) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ \vec{x} \\ \vec{x}^2 \\ \vec{x}^3 \\ \vec{x}^4 \\ \vdots \end{bmatrix} \tag{11.85}$$

The solution is non-trivial.

11.4.2 Derivatives at Three Points

Two base points and a number of derivatives at the two base points and the interpolation point are determined.

$$\partial f[0]\left(\vec{U}\right) = Y; \qquad \partial f[j]\left(\vec{U}\right) = (-1)^{j-1} * (j-1)! \; / \; \vec{U}^j; \qquad j > 0 \tag{11.86}$$

$$\partial f[j](\vec{x}) = (-1)^{j-1} * (j-1)! \; / \; \vec{x}^j; \qquad j > 0 \tag{11.87}$$

$$\partial f[0]\left(\vec{V}\right) = Z; \qquad \partial f[j]\left(\vec{V}\right) = (-1)^{j-1} * (j-1)! \; / \; \vec{V}^j; \qquad j > 0 \tag{11.88}$$

The number of conditions N is determined by the number of derivatives of first or higher order per point M.

$$N = 3 * M + 2 \tag{11.89}$$

The transposed polynomial is determined.

$$f(\vec{x}) = \left\{ \begin{array}{l} w[0](\vec{x}) * Y \\ +w[2 * M + 1](\vec{x}) * Z \end{array} \right\} + \sum^{1 \le j \le M} \left\{ \begin{array}{l} w[j](\vec{x}) * \partial f[j]\left(\vec{U}\right) \\ +w[M + 1 + j](\vec{x}) * \partial f[j](\vec{x}) \\ +w[2 * M + 2 + j](\vec{x}) * \partial f[j]\left(\vec{V}\right) \end{array} \right\} \tag{11.90}$$

The base polynomials are determined by a system of linear equations.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \dots \\ \vec{U} & \vec{V} & 1 & 1 & 1 & \dots \\ \vec{U}^2 & \vec{V}^2 & 2*\vec{U} & 2*\vec{x} & 2*\vec{V} & \dots \\ \vec{U}^3 & \vec{V}^3 & 3*\vec{U}^2 & 3*\vec{x}^2 & 3*\vec{V}^2 & \dots \\ \vec{U}^4 & \vec{V}^4 & 4*\vec{U}^3 & 4*\vec{x}^3 & 4*\vec{V}^3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} * \begin{bmatrix} w[1][0](\vec{x}) \\ w[1][1](\vec{x}) \\ w[1][2](\vec{x}) \\ w[1][3](\vec{x}) \\ w[1][4](\vec{x}) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ \vec{x} \\ \vec{x}^2 \\ \vec{x}^3 \\ \vec{x}^4 \\ \vdots \end{bmatrix} \tag{11.91}$$

The solution is non-trivial.

Chapter 12

Arc Tangent Functions

A number of functions of one dimension are discussed.

$$\text{size}\left(\vec{x}\right) = \text{size}\left(\vec{u}\right) = \text{size}\left(\mathbf{1}\right) = 1 \tag{12.1a}$$

The inverse trigonometric tangent functions are defined by their first derivative. The derivatives are different.

$$\partial\left(\mathbf{1}; \arctan(\vec{x})\,; \vec{u}\right) = \frac{1}{1+\vec{u}^2} = \frac{1}{\vec{u}^2+1} \tag{12.1b}$$

$$\partial\left(\mathbf{1}; \text{arccot}(\vec{x})\,; \vec{u}\right) = \frac{1}{-1-\vec{u}^2} = \frac{1}{-\vec{u}^2-1} \tag{12.1c}$$

The inverse hyperbolic tangent functions are defined by their first derivative. The derivatives are equal and defined for different domains.

$$\partial\left(\mathbf{1}; \text{artanh}(\vec{x})\,; \vec{u}\right) = \frac{1}{1-\vec{u}^2} = \frac{1}{-\vec{u}^2+1}; \qquad \text{abs}(\vec{u}) < 1 \tag{12.1d}$$

$$\partial\left(\mathbf{1}; \text{arcoth}(\vec{x})\,; \vec{u}\right) = \frac{1}{1-\vec{u}^2} = \frac{1}{-\vec{u}^2+1}; \qquad \text{abs}(\vec{u}) > 1 \tag{12.1e}$$

The combinations of signs gives another function of which the integration results the negative inverse hyperbolic tangent functions.

$$\partial\left(\mathbf{1}; -\text{artanh}(\vec{x})\,; \vec{u}\right) = \frac{1}{-1+\vec{u}^2} = \frac{1}{\vec{u}^2-1}; \qquad \text{abs}(\vec{u}) < 1 \tag{12.1f}$$

$$\partial\left(\mathbf{1}; -\text{arcoth}(\vec{x})\,; \vec{u}\right) = \frac{1}{-1+\vec{u}^2} = \frac{1}{\vec{u}^2-1}; \qquad \text{abs}(\vec{u}) > 1 \tag{12.1g}$$

12.1 Arc Tangent

The arc tangent is the unknown integral of (12.1b). The integral is approximated by integration of the two possible polynomial divisions of the first derivative.

The division by smallest orders is determined. It results a polynomial $p(\vec{u})$ and a rational remainder

$q(\vec{u})$.

$$\frac{1}{1+\vec{u}^2} = \sum^{0\le i<n} \left\{(-1)^i * \vec{u}^{2*i}\right\} + (-1)^n * \frac{\vec{u}^{2*n}}{1+\vec{u}^2} = p(\vec{x}) + q(\vec{x}) \tag{12.2}$$

The polynomial part is strictly stable within half the square root of Two and converges to the first derivative of the arc tangent.

$$\text{abs}\big((-1)^i * \vec{u}^{2*i}\big) > 2 * \text{abs}\Big((-1)^{i+1} * \vec{u}^{2*(i+1)}\Big) \tag{12.3a}$$

$$\text{if}\left(\text{abs}(\vec{u}) < \frac{1}{2} * \sqrt{2}\right) \text{then}\,(p(\vec{x}) \ \text{ is strictly stable}) \tag{12.3b}$$

Powers of exponents greater than One and bases greater than $\pm\sqrt{2}/2$ are large and a series of such a number is not determined by terms of small orders.

$$\frac{1}{1+\vec{u}^2} \approx \sum^{0\le i<n} \left\{(-1)^i * \vec{u}^{2*i}\right\}; \qquad \text{abs}(\vec{u}) < \frac{1}{2} * \sqrt{2} \tag{12.4}$$

The integral through the origin is an approximation of the arc tangent within a radius of $\sqrt{2}/2$.

$$\arctan(\vec{x}) \approx \sum^{0\le i<n} \left\{(-1)^i * \frac{\vec{x}^{2*i+1}}{2*i+1}\right\}; \qquad \text{abs}(\vec{u}) < \frac{1}{2} * \sqrt{2} \tag{12.5}$$

The division by largest orders is determined. It results a series of rational terms $P(\vec{x})$ and a rational remainder $Q(\vec{x})$.

$$\frac{1}{\vec{u}^2+1} = \sum^{1\le i\le n} \left\{\frac{(-1)^i}{\vec{u}^{2*i}}\right\} + \frac{(-1)^n}{\vec{u}^{2*n} * (\vec{u}^2+1)} = P(\vec{x}) + Q(\vec{x}) \tag{12.6}$$

The rational series is strictly stable outside a radius of the square root of Two.

$$\text{abs}\left(\frac{(-1)^i}{\vec{u}^{2*i}}\right) > 2 * \text{abs}\left(\frac{(-1)^{i+1}}{\vec{u}^{2*(i+1)}}\right) \tag{12.7}$$

$$\text{if}\left(\text{abs}(\vec{u}) > \sqrt{2}\right) \text{then}\,(P(\vec{x}) \ \text{ is strictly stable}) \tag{12.8}$$

The series converges to the first derivative of the arc tangent.

$$\frac{1}{\vec{u}^2+1} \approx \sum^{0\le i<n} \left\{\frac{(-1)^i}{\vec{u}^{2*i}}\right\}; \qquad \text{abs}(\vec{u}) > \sqrt{2} \tag{12.9}$$

The indefinite integral is determined.

$$f(\vec{x}) \approx A + \sum^{0\le i<n} \left\{\frac{(-1)^{i+1}}{(2*i+1) * \vec{x}^{2*i+1}}\right\} = A + g(\vec{x}) \tag{12.10}$$

The variable terms of the integral $g(\vec{x})$ tend to zero for large values of $\text{abs}(\vec{x})$.

$$\lim_{\text{abs}(\vec{x})\to\infty} g(\vec{x}) = 0 \tag{12.11}$$

$$\text{if}\left(\vec{x} < -\sqrt{2}\right) \text{then}\,(0 < g(\vec{x}) < 1) \tag{12.12}$$

$$\text{if}\left(\vec{x} > \sqrt{2}\right) \text{then}\,(-1 < g(\vec{x}) < 0) \tag{12.13}$$

The range of the arc tangent is defined.

$$-\frac{\pi}{2} < \arctan(\vec{x}) < \frac{\pi}{2} \tag{12.14}$$

Two definite integrals are defined in order to approximate the arc tangent for large negative and positive numbers. The integrals differ by the sign of the integration constant.

$$\arctan(\vec{x}) \approx \pm\frac{\pi}{2} + \sum^{0\leq i<n} \left\{ \frac{(-1)^{i+1}}{(2*i+1)*\vec{x}^{2*i+1}} \right\}; \qquad \pm\vec{x} > \sqrt{2} \tag{12.15}$$

Two domains are not defined by extrapolations if strict stability applies.

$$-\sqrt{2} \leq \vec{x} \leq -\frac{1}{2}*\sqrt{2}; \qquad \frac{1}{2}*\sqrt{2} \leq \vec{x} \leq \sqrt{2} \tag{12.16}$$

These domains may be defined by interpolation at one or more base points and one or more initial derivatives. For instance an interpolation may be defined by three base points and two initial derivatives each.

$$\sum^{0\leq i<3} \left\langle y[0][i] = \frac{1}{1+\vec{u}[i]} \right\rangle; \qquad \sum^{0\leq i<3} \left\langle y[1][i] = -\frac{2*\vec{u}}{(1+\vec{u}[i])^2} \right\rangle \tag{12.17a}$$

The base polynomials are determined by a system of linear equations.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ \vec{x}[0] & 1 & \vec{x}[1] & 1 & \vec{x}[2] & 1 \\ \vec{x}[0]^2 & 2*\vec{x}[0] & \vec{x}[1]^2 & 2*\vec{x}[1] & \vec{x}[2]^2 & 2*\vec{x}[2] \\ \vec{x}[0]^3 & 3*\vec{x}[0]^2 & \vec{x}[1]^3 & 3*\vec{x}[1]^2 & \vec{x}[2]^3 & 3*\vec{x}[2]^2 \\ \vec{x}[0]^4 & 4*\vec{x}[0]^3 & \vec{x}[1]^4 & 4*\vec{x}[1]^3 & \vec{x}[2]^4 & 4*\vec{x}[2]^3 \\ \vec{x}[0]^5 & 5*\vec{x}[0]^4 & \vec{x}[1]^5 & 5*\vec{x}[1]^4 & \vec{x}[2]^5 & 5*\vec{x}[2]^4 \end{bmatrix} * \begin{bmatrix} w[0](\vec{x}) \\ w[1](\vec{x}) \\ w[2](\vec{x}) \\ w[3](\vec{x}) \\ w[4](\vec{x}) \\ w[5](\vec{x}) \end{bmatrix} = \begin{bmatrix} 1 \\ \vec{x} \\ \vec{x}^2 \\ \vec{x}^3 \\ \vec{x}^4 \\ \vec{x}^5 \end{bmatrix} \tag{12.17b}$$

The tranposed polynomial is determined. The interpolation of the arc tangent is determined by the integral. The integration constant equals Zero.

$$h(\vec{u}) = \sum^{0\leq i<3} \{w[2*i](\vec{x}) * y[0][i] + w[2*i+1](\vec{x}) * y[1][i]\} \tag{12.17c}$$

12.2 Arc Cotangent

The arc cotangent is the unknown integral of (12.1c). The integral is approximated by integration of the two possible polynomial divisions of the first derivative.

The division by smallest orders is determined. It results a polynomial $p(\vec{u})$ and a rational reminder $q(\vec{u})$.

$$\frac{1}{-1-\vec{u}^2} = \sum^{0\leq i<n} \left\{(-1)^{i+1} * \vec{u}^{2*i}\right\} + (-1)^{n+1} * \frac{\vec{u}^{2*n}}{1+\vec{u}^2} = p(\vec{x}) + q(\vec{x}) \tag{12.18}$$

Strict stability applies according to (12.3a). The integral is an approximation of the arc cotangent within a radius of $\sqrt{2}/2$. The integration constant is defined by $\pi/2$.

$$\operatorname{arccot}(\vec{x}) \approx \frac{\pi}{2} + \sum^{0 \le i < n} \left\{ (-1)^{i+1} * \frac{\vec{x}^{2*i+1}}{2*i+1} \right\}; \qquad \operatorname{abs}(\vec{u}) < \frac{1}{2} * \sqrt{2} \tag{12.19}$$

The division by largest orders is determined. It results a series of rational terms $P(\vec{x})$ and a rational remainder $Q(\vec{x})$.

$$\frac{1}{-\vec{u}^2 - 1} = \sum^{1 \le i \le n} \left\{ \frac{(-1)^{i+1}}{\vec{u}^{2*i}} \right\} + \frac{(-1)^{n+1}}{\vec{u}^{2*n} * (\vec{u}^2 + 1)} = P(\vec{x}) + Q(\vec{x}) \tag{12.20}$$

Strict stability applies according to (12.7). The series converges to the first derivative of the arc cotangent outside the radius of strict stability.

$$\frac{1}{-\vec{u}^2 - 1} \approx \sum^{0 \le i < n} \left\{ \frac{(-1)^{i+1}}{\vec{u}^{2*i}} \right\}; \qquad \operatorname{abs}(\vec{u}) > \sqrt{2} \tag{12.21}$$

The range of the arc cotangent is defined.

$$0 < \operatorname{arccot}(\vec{x}) < \pi \tag{12.22}$$

Two definite integrals are determined in order to approximate the arc cotangent for large negative and positive numbers.

$$\operatorname{arccot}(\vec{x}) \approx \pi + \sum^{0 \le i < n} \left\{ \frac{(-1)^{i+1}}{\vec{x}^{2*i+1}} \right\}; \qquad \vec{x} < -\sqrt{2} \tag{12.23}$$

$$\arctan(\vec{x}) \approx \sum^{0 \le i < n} \left\{ \frac{(-1)^{i+1}}{\vec{x}^{2*i+1}} \right\}; \qquad \vec{x} > \sqrt{2} \tag{12.24}$$

Two domains are not defined by extrapolations.

$$-\sqrt{2} \le \vec{x} \le -\frac{1}{2} * \sqrt{2}; \qquad \frac{1}{2} \le \vec{x} \le \sqrt{2} \tag{12.25}$$

An identity applies.

$$\operatorname{arccot}(\vec{x}) = \frac{\pi}{2} - \arctan(\vec{x}) \tag{12.26}$$

12.3 Hyperbolic Arc Tangent and Cotangent

The hyperbolic arc tangent and cotangent are two parts of the unknown integral of (12.1d) or (12.1e).

The division by smallest orders is determined. It results a polynomial $p(\vec{u})$ and a rational remainder $q(\vec{u})$.

$$\frac{1}{1 - \vec{u}^2} = \sum^{0 \le i < n} \left\{ \vec{u}^{2*i} \right\} + \frac{\vec{u}^{2*n}}{1 - \vec{u}^2} = p(\vec{x}) + q(\vec{x}) \tag{12.27}$$

Strict stability applies according to (12.3a). The integral through the origin is an approximation of the hyperbolic arc tangent within a radius of $\sqrt{2}/2$.

$$\operatorname{artanh}(\vec{x}) \approx \sum^{0 \le i < n} \left\{ \frac{\vec{x}^{2*i+1}}{2*i+1} \right\}; \qquad \operatorname{abs}(\vec{u}) < \frac{1}{2} * \sqrt{2} \tag{12.28}$$

The division by largest orders is determined. It results a series of rational terms $P(\vec{x})$ and a rational remainder $Q(\vec{x})$.

$$\frac{1}{-\vec{u}^2+1} = \sum^{1 \le i \le n} \left\{ \frac{-1}{\vec{u}^{2*i}} \right\} + \frac{-1}{\vec{u}^{2*n} * (-\vec{u}^2+1)} = P(\vec{x}) + Q(\vec{x}) \tag{12.29}$$

Strict stability applies according to (12.7). The series converges to the first derivative of the hyperbolic arc cotangent outside the radius of strict stability.

$$\frac{1}{-\vec{u}^2+1} \approx \sum^{0 \le i < n} \left\{ \frac{-1}{\vec{u}^{2*i}} \right\}; \qquad \operatorname{abs}(\vec{u}) > \sqrt{2} \tag{12.30}$$

The hyperbolic arc cotangent is determined by one definite integral within the domain of strict stability.

$$\operatorname{arcoth}(\vec{x}) \approx \sum^{0 \le i < n} \left\{ \frac{1}{\vec{x}^{2*i+1}} \right\}; \qquad \operatorname{abs}(\vec{x}) > \sqrt{2} \tag{12.31}$$

Two domains are not defined by extrapolations.

$$-\sqrt{2} \le \vec{x} \le -\frac{1}{2} * \sqrt{2}; \qquad \frac{1}{2} \le \vec{x} \le \sqrt{2} \tag{12.32}$$

Chapter 13

Number Series

A polynomial of a one-dimensional position is determined by coefficients of One and a regular series of tupels.

$$f[a][b](\vec{u}) = \vec{u}^{a+0*b} \pm \vec{u}^{a+1*b} + \vec{u}^{a+2*b} \pm ... + (-1)^{n-1} * \vec{u}^{a+(n-1)*b} \tag{13.1}$$

$$= \vec{u}^{a} + \vec{u}^{b} * \left(\vec{u}^{a} + \vec{u}^{b} * \left(\vec{u}^{a} + ... * \left(\vec{u}^{a} + \vec{u}^{b} * \left(\vec{u}^{a} + \vec{u}^{b+a}\right)\right)\right)\right) \tag{13.2}$$

$$= \sum^{0 \le i < n} \left\{(-1)^{i} * \vec{u}^{a+i*b}\right\}; \qquad \langle a; b; i\rangle \in \mathbb{N};\ b > 0 \tag{13.3}$$

The polynomial may be determined by a pattern.

$$y[0] = \vec{u}^{a} - \vec{u}^{b+a} \tag{13.4a}$$

$$y[1] = \vec{u}^{a} - \vec{u}^{b} * y[0] \tag{13.4b}$$

$$y[2] = \vec{u}^{a} - \vec{u}^{b} * y[1] \tag{13.4c}$$

$$y[i+1] = \vec{u}^{a} - \vec{u}^{b} * y[i] \tag{13.4d}$$

The number series of Two is determined.

$$\lim_{n\to\infty} \left(f[0][1]\left(\overrightarrow{1/2}\right) = \sum^{0 \le i < N} \left\{\overrightarrow{(1/2)^{i}}\right\} \right) \to 2 \tag{13.5}$$

The integration polynomial is determined by an integration of Zero.

$$f[a][b](\vec{u}) = \partial\left(0; g[a][b](\vec{x}); \vec{u}\right) \tag{13.6a}$$

$$g[a][b](\vec{x}) = \frac{\vec{x}^{a+0*b+1}}{a+0*b+1} \pm \frac{\vec{x}^{a+1*b+1}}{a+1*b+1} + \frac{\vec{x}^{a+2*b+1}}{a+2*b+1} \pm ... + \frac{(-1)^{n-1} * \vec{x}^{a+(n-1)*b+1}}{a+(n-1)*b+1} \tag{13.6b}$$

$$= \sum^{0 \le i < n} \left\{\frac{(-1)^{i} * \vec{x}^{a+i*b+1}}{a+i*b+1}\right\}; \qquad \langle a; b; i\rangle \in \mathbb{N};\ b > 0 \tag{13.6c}$$

Two number series are determined.

$$a = 1;\ b = 0;\ \vec{x} = 1; \qquad \lim_{n\to\infty}\left(\sum^{0\le i<n}\left\{\frac{(-1)^i}{i+1}\right\}\right) \to \log[\mathsf{e}](2) \tag{13.7}$$

$$a = 2;\ b = 0;\ \vec{x} = 1; \qquad \lim_{n\to\infty}\left(\sum^{0\le i<n}\left\{\frac{(-1)^i}{2*i+1}\right\}\right) \to \frac{\pi}{4} \tag{13.8}$$

The square of the integration polynomial is determined.

$$q[a][b](\vec{x}) = \sum^{0\le j<n}\left\{\frac{(-1)^j * \vec{x}^{a+j*b+1}}{a+j*b+1}\right\} * \sum^{0\le i<n}\left\{\frac{(-1)^i * \vec{x}^{a+i*b+1}}{a+i*b+1}\right\}; \qquad i\in\mathbb{N};\ j\in\mathbb{N} \tag{13.9}$$

The square contains the pattern of a diagonal of index e.

$$p[c][d][e](\vec{x}) = \sum^{0\le k<m}\left\{\frac{(-1)^k * \vec{x}^{c+k*d+1}}{c+k*d+1} * \frac{(-1)^k * \vec{x}^{c+k*d+1+d*e}}{c+k*d+1+d*e}\right\}; \qquad i\in\mathbb{N} \tag{13.10}$$

A diagonal at position One is determined. The signs cancel.

$$p[c][d][e]\left(\vec{1}\right) = \sum^{0\le k<n-b}\left\{\frac{1}{c+k*d+1} * \frac{1}{c+k*d+1+d*e}\right\}; \qquad i\in\mathbb{N} \tag{13.11}$$

The limit of the first secondary diagonal with offset Zero is determined.

$$\lim_{m\to\infty}\left(p[0][d][1]\left(\vec{1}\right) = \sum^{0\le k<n-d}\left\{\frac{1}{k*d+1} * \frac{1}{k*d+1+d}\right\}\right) \to \frac{1}{d} \tag{13.12}$$

Diagonal of Offset Zero and Step One

The limit of the primary diagonal is determined.

$$\lim_{m\to\infty}\left(p[0][1][0]\left(\vec{1}\right) = \sum^{0\le k<m}\left\{\frac{1*1}{(k+1)*(k+1)}\right\}\right) \to \frac{\pi^2}{6} \tag{13.13}$$

The limit of the first secondary diagonal is determined.

$$\lim_{m\to\infty}\left(p[0][1][1]\left(\vec{1}\right) = \sum^{0\le k<m}\left\{\frac{1*1}{(k+1)*(k+2)}\right\}\right) \to 1 \tag{13.14}$$

The limit of the second secondary diagonal is determined.

$$\lim_{m\to\infty}\left(p[0][1][2]\left(\vec{1}\right) = \sum^{0\le k<m}\left\{\frac{1*1}{(k+1)*(k+3)}\right\}\right) \to \frac{3}{4} \tag{13.15}$$

The limit of the third secondary diagonal is determined.

$$\lim_{m\to\infty}\left(p[0][1][3]\left(\vec{1}\right) = \sum^{0\le k<m}\left\{\frac{1*1}{(k+1)*(k+4)}\right\}\right) \to \frac{11}{18} \tag{13.16}$$

The initial half of the primary diagonal is determined. The limit is determined.

$$\lim_{n\to\infty}\left(\frac{1}{2}*\left(p[0][1][0]\left(\vec{1}\right)=\sum^{0\le k<m/2}\left\{\frac{1*1}{(k+1)*(k+1)}\right\}\right)\right)\to\frac{\pi^2}{12} \tag{13.17a}$$

The equation is modified.

$$A=\frac{2^2}{2}*\sum^{0\le k<m/2}\left\{\frac{1*1}{2^2*(k+1)*(k+1)}\right\} \tag{13.17b}$$

The difference of the primary diagonal and its modified initial half is determined.

$$p[0][1][0]\left(\vec{1}\right)-A=\sum^{0\le k<m}\left\{(-1)^{k+1}*\frac{1*1}{(k+1)*(k+1)}\right\} \tag{13.17c}$$

The limit of the difference is determined.

$$\lim_{m\to\infty}\left(\sum^{0\le k<m}\left\{(-1)^{k}*\frac{1*1}{(k+1)*(k+1)}\right\}\right)\to\frac{\pi^2}{12} \tag{13.17d}$$

The initial half of the primary diagonal is divided by Four. The limit is determined.

$$\lim_{m\to\infty}\left(\frac{1}{4}*\left(p[0][1][0]\left(\vec{1}\right)=\sum^{0\le k<m/2}\left\{\frac{1*1}{(k+1)*(k+1)}\right\}\right)\right)\to\frac{\pi^2}{24} \tag{13.18a}$$

The equation is modified.

$$B=\sum^{0\le k<m/2}\left\{\frac{1*1}{(2*k+2)*(2*k+2)}\right\} \tag{13.18b}$$

The difference of the primary diagonal and its modified initial half is determined.

$$p[0][1][0]\left(\vec{1}\right)-B=\sum^{0\le k<m}\left\{\frac{1*1}{(2*k+1)*(2*k+1)}\right\} \tag{13.18c}$$

The limit of the difference is determined.

$$\lim_{m\to\infty}\left(\sum^{0\le k<m}\left\{\frac{1*1}{(2*k+1)*(2*k+1)}\right\}\right)\to\frac{\pi^2}{8} \tag{13.18d}$$

Primary Diagonal of Offset Zero and Step Two

$$\lim_{m\to\infty}\left(p[0][2][0]\left(\vec{1}\right)=\sum^{0\le k<m}\left\{\frac{1*1}{(2*k+1)*(2*k+1)}\right\}\right)\to\frac{\pi^2}{8} \tag{13.19}$$

Second Secondary Diagonal of Offset Zero and Step Two

$$\lim_{m\to\infty}\left(p[0][2][2]\left(\vec{1}\right)=\sum^{0\le k<m}\left\{\frac{1*1}{(2*k+1)*(2*k+5)}\right\}\right)\to\frac{1}{3} \tag{13.20}$$

Third Secondary Diagonal of Offset Zero and Step Two

$$\lim_{m\to\infty}\left(p[0][2][3]\left(\vec{1}\right)=\sum^{0\le k<m}\left\{\frac{1*1}{(2*k+1)*(2*k+7)}\right\}\right)\to\frac{23}{90} \tag{13.21}$$

Primary Diagonal of Offset One and Step Two

$$\lim_{m\to\infty}\left(p[1][2][0]\left(\vec{1}\right)=\sum^{0\le k<m}\left\{\frac{1*1}{(2*k+2)*(2*k+2)}\right\}\right)\to\frac{\pi^2}{24} \tag{13.22}$$

First Secondary Diagonal of Offset One and Step Two

$$\lim_{m\to\infty}\left(p[1][2][1]\left(\vec{1}\right)=\sum^{0\le k<m}\left\{\frac{1*1}{(2*k+2)*(2*k+4)}\right\}\right)\to\frac{1}{4} \tag{13.23}$$

Second Secondary Diagonal of Offset One and Step Two

$$\lim_{m\to\infty}\left(p[1][2][2]\left(\vec{1}\right)=\sum^{0\le k<m}\left\{\frac{1*1}{(2*k+2)*(2*k+6)}\right\}\right)\to\frac{3}{16} \tag{13.24}$$

Third Secondary Diagonal of Offset One and Step Two

$$\lim_{m\to\infty}\left(p[1][2][3]\left(\vec{1}\right)=\sum^{0\le k<m}\left\{\frac{1*1}{(2*k+2)*(2*k+8)}\right\}\right)\to\frac{11}{72} \tag{13.25}$$

Second Secondary Diagonal of Offset Zero and Step Three

$$\lim_{m\to\infty}\left(p[0][3][2]\left(\vec{1}\right)=\sum^{0\le k<m}\left\{\frac{1*1}{(3*k+1)*(3*k+7)}\right\}\right)\to\frac{5}{24} \tag{13.26}$$

First Secondary Diagonal of Offset One and Step Three

$$\lim_{m\to\infty}\left(p[1][3][1]\left(\vec{1}\right)=\sum^{0\le k<m}\left\{\frac{1*1}{(3*k+2)*(3*k+5)}\right\}\right)\to\frac{1}{6} \tag{13.27}$$

Second Secondary Diagonal of Offset One and Step Three

$$\lim_{m\to\infty}\left(p[1][3][2]\left(\vec{1}\right)=\sum^{0\le k<m}\left\{\frac{1*1}{(3*k+2)*(3*k+8)}\right\}\right)\to\frac{7}{60} \tag{13.28}$$

Primary Diagonal of Offset Two and Step Three

$$\lim_{m\to\infty}\left(p[2][3][0]\left(\vec{1}\right) = \sum^{0\le k<m}\left\{\frac{1*1}{(3*k+3)*(3*k+3)}\right\}\right) \to \frac{\pi^2}{54} \tag{13.29}$$

First Secondary Diagonal of Offset Two and Step Three

$$\lim_{m\to\infty}\left(p[2][3][1]\left(\vec{1}\right) = \sum^{0\le k<m}\left\{\frac{1*1}{(3*k+3)*(3*k+6)}\right\}\right) \to \frac{1}{9} \tag{13.30}$$

Second Secondary Diagonal of Offset Two and Step Three

$$\lim_{m\to\infty}\left(p[2][3][2]\left(\vec{1}\right) = \sum^{0\le k<m}\left\{\frac{1*1}{(3*k+3)*(3*k+9)}\right\}\right) \to \frac{1}{12} \tag{13.31}$$

Second Secondary Diagonal of Offset Zero and Step Four

$$\lim_{m\to\infty}\left(p[0][4][2]\left(\vec{1}\right) = \sum^{0\le k<m}\left\{\frac{1*1}{(4*k+1)*(4*k+9)}\right\}\right) \to \frac{3}{20} \tag{13.32}$$

Listing 13.1: Values of Diagonals in C with gmp [4]

```
#include <math.h>
#include <stdio.h>
#include <gmp.h>

int main(void)
{
  unsigned c, d, e, k;
  mpf_t p, s, t;
  //
  mpf_set_default_prec(2<<14);
  mpf_init(p); mpf_init(s); mpf_init(t);
  //
  printf("(c + d*i + 1)*(c + d*i + 1 + d*e)\n");
  // for each step
  for(d = 0; d < 5; ++d)
  {
    // for each offset
    for(c = 0; c < d; ++c)
    {
      // for each diagonal
      for(e = 0; e < 3; ++e)
      {
        mpf_set_ui(s, 0);
        for(k = 0; k < 50000; ++k)
        {
          // start 1 because of integration
          mpf_set_ui(p, d*k + c + 1);
          mpf_mul_ui(p, p, d*k + c + 1 + d*e);
          mpf_set_ui(t, 1);
          mpf_div(t, t, p);
          mpf_add(s, s, t);
        }
        printf("c=%d, d=%d, e=%d, (%d*k+%d)*(%d*k+%d) = ",
          c, d, e, d, c + 1, d, c + 1 + d*e);
        mpf_out_str(stdout, 10, 8, s);
        printf("\n");
        mpf_set_ui(t, 0);
      }
    }
    printf("\n");
  }
  //
  mpf_clear(p); mpf_clear(s); mpf_clear(t);
  return 0;
}
```

Chapter 14

Heat Equation of One Spatial Dimension

Solutions of problems on domains of two dimensions are analytically complicated. The solution to the heat equation of one spatial and one temporal dimension on a uniform grid is given as one of the simpler of these problems.

A value is determined by a polynomial operator.

$$f\left(\overrightarrow{\langle t; x\rangle}\right) = a + b * t + c * x + d * x^2 \tag{14.1}$$

The value of the polynomial at a next point in time is determined by three Dirichlet conditions and the heat condition of a diffusion coefficient d.

$$\partial f[\mathbf{0}]\left(\overrightarrow{\langle 0; -\Delta x\rangle}\right) = y[l] \tag{14.2}$$

$$\partial f[\mathbf{0}]\left(\vec{\mathbf{0}}\right) = y\left[\vec{\mathbf{0}}\right] \tag{14.3}$$

$$\partial f[\langle 1; 0\rangle]\left(\vec{\mathbf{0}}\right) - d * \partial f[\langle 0; 2\rangle]\left(\vec{\mathbf{0}}\right) = 0 \tag{14.4}$$

$$\partial f[\mathbf{0}]\left(\overrightarrow{\langle 0; \Delta x\rangle}\right) = y[r] \tag{14.5}$$

The base polynomials are determined by a system of linear equations.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -\Delta x & 0 & 0 & \Delta x \\ \Delta x^2 & 0 & -2 * d & \Delta x^2 \end{bmatrix} * \begin{bmatrix} w[l] \\ w\left[\vec{\mathbf{0}}\right] \\ w[c] \\ w[r] \end{bmatrix} = \begin{bmatrix} 1 \\ \Delta t \\ 0 \\ 0 \end{bmatrix} \tag{14.6}$$

The base polynomials to the conditions are determined.

$$w[l] = d * \frac{\Delta t}{\Delta x^2} = B; \quad w\left[\vec{\mathbf{0}}\right] = 1 - 2 * d * \frac{\Delta t}{\Delta x^2} = A; \quad w[c] = \Delta t; \quad w[r] = d * \frac{\Delta t}{\Delta x^2} = B \tag{14.7}$$

It is assumed that the time step is determined by the base polynomial or weight of the Dirichlet condition at the origin.

$$\Delta t < \frac{\Delta x^2}{2 * d} \tag{14.8}$$

The value of the polynomial operator at a next point is determined by base polynomials.

$$f\left(\overrightarrow{\langle \Delta t; 0 \rangle}\right) = y[\Delta t] = B * y[l] + A * y\left[\vec{\mathbf{0}}\right] + B * y[r] \tag{14.9}$$

A domain of n points in space and m lines in time is determined. The boundary values in time are different constants.

$$\sum^{0 \le j < m} \left\langle \begin{array}{r} Y[j][0] = \text{const} \\ Y[j][n-1] = \text{const} \end{array} \right\rangle \tag{14.10}$$

The initial values in space are different constants.

$$\sum^{1 \le i < n-1} \langle Y[0][i] = \text{const} \rangle \tag{14.11}$$

The variables in space and time are determined.

$$\sum^{0 \le j < m} \left\langle \sum^{1 \le i < n-1} \langle Y[j+1][i] = B * Y[j][i-1] + A * Y[j][i] + B * Y[j][i+1] \rangle \right\rangle \tag{14.12}$$

The solution at a point 0 depends on a number of points 1.

$$p[0] = \left\langle \vec{X}[J[0]][I[0]]\,; Y[J[0]][I[0]] \right\rangle; \quad p[1] = \left\langle \vec{X}[J[1]][I[1]]\,; Y[J[1]][I[1]] \right\rangle; \quad J[0] > J[1] \tag{14.13}$$

The difference of the spatial indexes is determined.

$$D = \text{abs}(I[0] - I[1]) \tag{14.14}$$

The difference of the smaller index to the index of the left boundary is determined.

$$L = \min\left(I[0]\,; I[1]\right) \tag{14.15}$$

The difference of the greater index to the index of the right boundary is determined.

$$R = n - 1 - \max\left(I[0]\,; I[1]\right) \tag{14.16}$$

The number of differences is determined.

$$W = L + D + R = N - 1 \tag{14.17}$$

The difference of the temporal indexes is determined.

$$T = J[0] - J[1] \tag{14.18}$$

The general part is determined by a single sum.

$$u(C) = \sum^{0 \le i \le (T-d)/2} \left\{ \binom{T}{C+i} * \binom{T-C-i}{i} * A^{T-C-2*i} * B^{C+2*i} \right\} \tag{14.19}$$

The specific part is determined by a double sum.

$$v(C) = \sum^{0 \le j \le \frac{T-C}{2*W}} \left\{ \sum^{0 \le i \le \frac{T-2*j*W-C}{2}} \left\{ \begin{array}{l} \binom{T}{2*j*W+C+i} * A^{T-2*j*W-C-2*i} \\ *\binom{T-2*j*W-C-i}{i} * B^{2*j*W+C+2*i} \end{array} \right\} \right\} \tag{14.20}$$

The part of point 1 on the solution at point 0 is determined.

$$Z[\langle J[0]\,;I[0]\rangle][\langle J[1]\,;I[1]\rangle] = Y[J[1]][I[1]] * \left(\begin{array}{l} u(D) \\ -v(2*L+D) + v(2*R+D) \\ -v(2*W+D) - v(2*W-D) \end{array} \right) \tag{14.21}$$

The solution at point 0 is determined.

$$Y[j][i] = \sum^{0 \le k < j} \{Z[\langle j;i\rangle][\langle k;0\rangle] + Z[\langle j;i\rangle][\langle k;n-1\rangle]\} + \sum^{1 \le k < n-1} \{Z[\langle j;i\rangle][\langle 0;k\rangle]\} \tag{14.22}$$

Neither the general nor the specific part changes signs. Thus the solution cannot be expressed in terms of sines. An expression in terms of other analytical functions is unknown.

Bibliography

[1] http://maxima.sourceforge.net, visited 5. October 2010

[2] Dictionary of Mathematics, David Nelson, Penguin Books, 2003

[3] Taschenbuch Mathematischer Formeln, Hans-Jochen Bartsch, Fachbuchverlag Leipzig im Carl Hanser Verlag, 21. Auflage, 2007

[4] http://gmplib.org, visited 15. October 2010

[5] http://www.gnu.org/software/gsl, visited 29. October 2010