Logarithm

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A conditionally convergent series of the natural logarithm is derived for its entire domain.

The natural logarithm is the unknown integral of a hyperbola.

$$y = \ln(x); \qquad \frac{d}{dx}\ln(x) = \frac{1}{x}; \qquad x > 0 \tag{1}$$

Derivatives of higher order follow accordingly.

$$\frac{d^j}{dx^j}\ln(x) = (-1)^{j-1} * \frac{(j-1)!}{U^j} \tag{2}$$

Natural logarithm and exponential function are inverse.

$$\ln(\mathsf{e}^x) = x \tag{3}$$

Logarithms of another base than e are multiples of the natural logarithm.

$$b^y = x; \qquad y = \log_b(x) = \frac{\ln(x)}{\ln(b)} \tag{4}$$

The logarithm is approximated by a polynomial.

$$f(x) = \sum^{0 \le i < n} a_i * x^i \tag{5}$$

The polynomial is to equate a point of the logarithm and a number of derivatives at that point.

$$f(U) = \frac{d^0 f(U)}{dx^0} = \ln(U) = Y; \qquad \frac{d^j f(U)}{dx^j} = (-1)^{j-1} * \frac{(j-1)!}{U^j}; \qquad j > 0 \tag{6}$$

Each condition is scaled by a weight w_i. A sum of all weighted conditions is determined.

$$w_0 * f(U) + \sum^{1 \le j < n} w_j * \frac{d^j f(U)}{dx^j} = w_0 * Y + \sum^{1 \le j < n} w_j * (-1)^{j-1} * \frac{(j-1)!}{U^j} \tag{7}$$

Suppose the weighted sum equals the polynomial.

$$f(x) = w_0 * f(U) + \sum^{1 \le j < n} w_j * \frac{d^j f(U)}{dx^j} \tag{8}$$

The derivatives of the polynomial are determined at the base point.

$$\begin{aligned}
f(x) &= a_0 + a_1 * x + a_2 * x^2 + a_3 * x^3 + a_4 * x^4 + a_5 * x^5 + ... && (9)\\
\frac{df(U)}{dx} &= a_1 + 2 * a_2 * U + 3 * a_3 * U^2 + 4 * a_4 * U^3 + 5 * a_5 * U^4 + ... && (10)\\
\frac{d^2 f(U)}{dx^2} &= 2 * a_2 + 6 * a_3 * U + 12 * a_4 * U^2 + 20 * a_5 * U^3 + ... && (11)\\
\frac{d^3 f(U)}{dx^3} &= 6 * a_3 + 24 * a_4 * U + 60 * a_5 * U^2 + ... && (12)\\
&... && (13)
\end{aligned}$$

The descending faculty is defined in order to express derivatives generally.

$$(a\text{¡}b) = \frac{a!}{(a-b)!} \tag{14}$$

A derivative of the polynomial is defined generally.

$$\frac{d^j f(U)}{dx^j} = \sum^{j \le i < n} a_i * (i\text{¡}j) * U^{i-j} \tag{15}$$

The weights are determined by a system of linear equations according to a comparison by the coefficients a_i.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ U & 1 & 0 & 0 & \dots \\ U^2 & 2 * U & 2 & 0 & \dots \\ U^3 & 3 * U^2 & 6 * U & 6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} * \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \end{bmatrix} \tag{16}$$

The base matrix is triangular such that the solution is available explicitly.

$$\begin{aligned}
w_0 &= 1 && (17)\\
w_1 &= x - U * w_0 && (18)\\
w_2 &= \frac{1}{2} * \left(x^2 - U^2 * w_0 - 2 * U * w_1\right) && (19)\\
w_3 &= \frac{1}{6} * \left(x^3 - U^3 * w_0 - 3 * U^2 * w_1 - 6 * U * w_2\right) && (20)\\
w_m &= \frac{1}{m!} * \left(x^m - \sum^{0 \le k < m} (m\text{¡}k) * U^{m-k} * w_k\right) && (21)
\end{aligned}$$

The weights are noted explicitly.

$$w_0 = 1 \tag{22}$$

$$w_1 = x - U \tag{23}$$

$$w_2 = \frac{1}{2!} * \left(x^2 - U^2 - 2 * U * (x - U)\right) = \frac{1}{2!} * (x - U)^2 \tag{24}$$

$$w_3 = \frac{1}{3!} * \left(x^3 - U^3 - 3 * U^2 * (x - U) - \frac{(3¡2)}{2!} * U * (x - U)^2\right) = \frac{1}{3!} * (x - U)^3 \tag{25}$$

$$w_m = \frac{1}{m!} * \left(x^m - \sum^{0 \le k < m} \binom{m}{k} * U^{m-k} * (x - U)^k\right) = \frac{1}{m!} * (x - U)^m \tag{26}$$

The value of the weights is substituted into the polynomial.

$$f(x) = f(U) + \sum^{1 \le j < n} \frac{1}{j!} * (x - U)^j * \frac{d^j f(U)}{dx^j} \tag{27}$$

The derivatives of the logarithm are noted explicitly.

$$f(x) = Y + \sum^{1 \le j < n} \frac{1}{j!} * (x - U)^j * (-1)^{j-1} * \frac{(j-1)!}{U^j} \tag{28}$$

A series is determined.

$$f(x) = Y + \sum^{1 \le j < n} (-1)^{j-1} * \frac{(x - U)^j}{j * U^j} \tag{29}$$

D'Alembert's convergence test of 1/2 is applied.

$$\frac{1}{2} * \left|\frac{(x - U)^j}{j * U^j}\right| > \left|\frac{(x - U)^{j+1}}{(j+1) * U^{j+1}}\right| \tag{30a}$$

$$\left|\frac{j+1}{j} * U\right| > 2 * |x - U| \tag{30b}$$

$$|U| > 2 * |x - U| \tag{30c}$$

The series converges conditionally.

$$\ln(x) = Y + \sum^{1 \le j < n} (-1)^{j-1} * \frac{(x - U)^j}{j * U^j}; \qquad U > 0; \quad U > 2 * |x - U|; \quad Y = \ln(U) \tag{31}$$

Base points may be determined by the exponential function.

$$\mathsf{e}^2 > 2 * \left|10 - \mathsf{e}^2\right|; \qquad f(10) \approx 2 + \sum^{1 \le j < 3} (-1)^{j-1} * \frac{\left(10 - \mathsf{e}^2\right)^j}{j * \mathsf{e}^{2*j}} \approx 2.305630 \tag{32}$$

$$\mathsf{e}^4 > 2 * \left|50 - \mathsf{e}^4\right|; \qquad f(50) \approx 4 + \sum^{1 \le j < 3} (-1)^{j-1} * \frac{\left(50 - \mathsf{e}^4\right)^j}{j * \mathsf{e}^{4*j}} \approx 3.912036 \tag{33}$$

References

[1] Logarithm, Hans-Dieter Reuter, http://www.joinedpolynomials.org/logarithm.pdf

[2] Joined Polynomials, Hans-Dieter Reuter, http://www.joinedpolynomials.org/jp.pdf