Poisson's Equation

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Version January 2, 2011
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Introduction

A solution to Poisson's Equation of one dimension is presented.

$$\frac{df(x)}{dx} = \text{const} \tag{1}$$

The domain is discretized by a number of equidistant points.

$$y_i = f(x_i); \qquad \frac{df(x_i)}{dx} = s_i \tag{2}$$

Poisson Operator

A local polynomial is assigned to each point.

$$f(h) = a^0 + a^1 * h + a^2 * h^2 \tag{3}$$

Poisson's equation is applied to each polynomial.

$$2 * a_2 = s_i \tag{4}$$

Adjacent polynomials are joined by Dirichlet conditions.

$$y_{i-1} = f(-h) = a^0 - a^1 * h + a^2 * h^2 \tag{5}$$

$$y_{i+1} = f(h) = a^0 + a^1 * h + a^2 * h^2 \tag{6}$$

The operator is determined by a transposition [2].

$$y_i = f(0) = w_0 * y_{i-1} + w_1 * s_i + w_2 * y_{i+1} \tag{7}$$

The weights are determined by a system of linear equations.

$$\begin{bmatrix} 1 & 0 & 1 \\ -h & 0 & h \\ h^2 & 2 & h^2 \end{bmatrix} * \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \qquad w = \left(\frac{1}{2}, -\frac{h^2}{2}, \frac{1}{2}\right) \tag{8}$$

A value is determined explicitly by a transposed local polynomial.

$$y_i = \frac{1}{2} * y_{i-1} - \frac{h^2}{2} * q_i + \frac{1}{2} * y_{i+1} \tag{9}$$

A value is determined implicitly by a Poisson Operator.

$$-y_{i-1} + 2 * y_i - y_{i+1} = -h^2 * s_i = q_i \tag{10}$$

System of Equations

A uniform tridiagonal square system of equations is determined by n equidistant base points. The bounds of the domain are contained in the sources.

$$D_n * y = \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & & . \; . \; . & & & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{bmatrix} * \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ . \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ . \\ q_{n-4} \\ q_{n-3} \\ q_{n-2} \end{bmatrix} \tag{11}$$

The determinant of a domain of n points is determined. The system consists of $n-2$ equations.

$$\det(D_n) = d_n = n - 1 \tag{12}$$

A source matrix of $n-2$ equations and the k-th column replaced is determined.

$$P_{n,k} = \begin{bmatrix} 2 & -1 & & & q_1 & & & & \\ -1 & 2 & -1 & & q_2 & & & & \\ & -1 & 2 & -1 & q_3 & & & & \\ & & & . & . & . & & & \\ & & & & q_{n-4} & -1 & 2 & -1 & \\ & & & & q_{n-3} & & -1 & 2 & -1 \\ & & & & q_{n-2} & & & -1 & 2 \end{bmatrix} \tag{13}$$

The determinant of a source matrix is determined.

$$\det(P_{n,k}) = p_{n,k} = d_{n-1-k} * \sum^{0 \le i < k} (q_{i+1} * D_{i+2}) + d_{k+2} * \sum^{k \le i < n-1} (q_{i+1} * D_{n-1-i}) \tag{14}$$

The solution to Poisson's equation is determined by Cramer's rule.

$$y_{k+1} = \frac{p_{n,k}}{d_n} = \frac{(n-2-k) * \sum^{0 \le i < k} (q_{i+1} * (i+1)) + (k+1) * \sum^{k \le i < n-2} (q_{i+1} * (n-2-i))}{n-1} \tag{15}$$

The solution to Laplace's equation is determined by the sources at the ends only.

$$y_{k+1} = \frac{p_{n,k}}{d_n} = \frac{(n-2-k) * q_1 + (k+1) * q_{n-2}}{n-1} \tag{16}$$

See [2] for an interpolation of the sine by this same method.

References

[1] Poisson's Equation, Hans-Dieter Reuter, http://www.joinedpolynomials.org/poisson.pdf

[2] Joined Polynomials, Hans-Dieter Reuter, http://www.joinedpolynomials.org/jp.pdf