Sine Theorem

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Introduction

This article shows how to express the sine exactly as a sum along the components of a Fibonacci number. The sum is derived by a recurrence relation on a sine operator.

Sine Operator

The sine operator is determined by two base points one left to and another at the origin and one condition of simple harmonic motion of a distribution coefficient c at the origin.

$$f(-H) = y_L; \qquad f(0) = y_0; \qquad c^2 * f(0) + \frac{d^2 f(0)}{dh^2} = 0; \qquad c > 0 \tag{1}$$

Three conditions determine a polynomial of three terms.

$$f(h) = a_0 * h^0 + a_1 * h^1 + a_2 * h^2; \qquad \frac{d^2 f(h)}{dh^2} = 2 * a_2 \tag{2}$$

Each condition is scaled by a weight w_i. A sum of the weighted conditions is determined.

$$w_L * \left(a_0 - a_1 * H + a_2 * H^2\right) + w_0 * a_0 + w_1 * \left(c^2 * a_0 + 2 * a_2\right) = w_L * y_L + w_0 * y_0 \tag{3}$$

The sum equals the polynomial under three conditions.

$$f(h) = w_L * y_L + w_0 * y_0; \qquad \begin{bmatrix} 1 & 1 & c^2 \\ -H & 0 & 0 \\ H^2 & 0 & 2 \end{bmatrix} * \begin{bmatrix} w_L \\ w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} 1 \\ h \\ h^2 \end{bmatrix} \tag{4}$$

The polynomial is determined by two weights.

$$w_L = -\frac{h}{H}; \qquad w_0 = -\frac{c^2 * h * H * (H + h) - 2 * H - 2 * h}{2 * H} \tag{5}$$

The solution simplifies if the constant equals the variable that is the distance H to the left equals the extrapolation to the right.

$$w_L = -1; \qquad w_0 = 2 - c^2 * h^2 \tag{6}$$

Analysis

Suppose the solution is a sine of a frequency d.

$$f(h) = R * \sin(\varphi + d * h) \tag{7}$$

The values and weights are substituted into the polynomial.

$$f(h) = w_L * y_L + w_0 * y_0 \tag{8}$$
$$R * \sin(\varphi + d * h) = R * \sin(\varphi - d * h) * w_L + R * \sin(\varphi) * w_0 \tag{9}$$
$$R * \sin(\varphi + d * h) = R * \sin(\varphi - d * h) * (-1) + R * \sin(\varphi) * \left(2 - c^2 * h^2\right) \tag{10}$$

A trigonometric addition formula applies.

$$\sin(a \pm b) = \sin(a) * \cos(b) \pm \cos(a) * \sin(b) \tag{11}$$

The formula is applied and two terms cancel. The scalar $R * \sin(\varphi)$ cancels. Note that w_L equals negative One.

$$\cos(d * h) = -\cos(d * h) + \left(2 - c^2 * h^2\right) \tag{12}$$

Distribution coefficient c and frequency d are not equal. However, the limit of the right hand side tends to c for small differences, see [2] for details.

$$d = \frac{1}{h} * \arccos\left(1 - \frac{1}{2} * c^2 * h^2\right); \qquad \lim_{h \to 0}\left(\frac{1}{h} * \arccos\left(1 - \frac{1}{2} * c^2 * h^2\right)\right) = c \tag{13}$$

The upper bound of difference h is determined by the domain of the arcus cosine.

$$\left|1 - \frac{1}{2} * c^2 * h^2\right| \le 1; \qquad c^2 * h^2 \le 2 \tag{14}$$

The polynomial is determined only if h is non-zero. Therefore the lower bound is excluded. The value of $\arccos(1)$ is zero and would result a difference of zero. Therefore the upper bound is excluded. The intersected domain is determined.

$$0 < h < \frac{\sqrt{2}}{c} \tag{15}$$

Sine Recurrence Relation

The sine recurrence relation is a numerical pattern that determines the sine. Values are repeatedly determined by two preceding values. These values are scaled by the same weights due to a uniform discretization.

$$y_h = y_L * w_L + y_0 * w_0 \tag{16}$$
$$y_{2h} = y_0 * w_L + y_h * w_0 \tag{17}$$
$$= y_0 * w_L + (y_L * w_L + y_0 * w_0) * w_0 \tag{18}$$
$$= y_L * w_L * w_0 + y_0 * \left(w_L + w_0^2\right) \tag{19}$$
$$y_{3h} = y_h * w_L + y_2 * h * w_0 \tag{20}$$
$$= y_L * \left(w_L^2 + w_L * w_0^2\right) + y_0 * \left(2 * w_L * w_0 + w_0^3\right) \tag{21}$$
$$\ldots$$

Each value of the right-hand-side is scaled by a composed weight in terms of a sum. The sum is similar to a binomial expansion but does not reduce to a basic operation.

$$W_{j,k} = \sum^{0 \le i \le \lfloor \frac{j-k}{2} \rfloor} \binom{j-k-i}{i} * w_L^{i+k} * w_0^{j-k-2*i} \tag{22}$$

$$= \sum^{0 \le i \le \lfloor \frac{j-k}{2} \rfloor} (-1)^{i+k} * \binom{j-k-i}{i} * w_0^{j-k-2*i} \tag{23}$$

The j-th value of the repetition is determined.

$$y_{j*h} = y_L * W_{j,1} + y_0 * W_{j,0} \tag{24}$$

The value at the origin y_0 equals zero such that the value to the right y_1 depends only on the value to the left y_L.

$$y_{j*h} = y_L * \sum^{0 \le i \le \lfloor \frac{j-1}{2} \rfloor} (-1)^{i+1} * \binom{j-1-i}{i} * w_0^{j-1-2*i} \tag{25}$$

The offset φ equals zero since y_0 equals zero. An offset of π or 180 deg is determined by the sign of y_L.

$$y_{j*h} = y_L * W_{j,1} \tag{26}$$

$$\sin(j * d * h) = \sin(-d * h) * W_{j,1} \tag{27}$$

The sine theorem is determined.

$$\frac{\sin(j * d * h)}{\sin(-d * h)} = \sum^{0 \le i \le \lfloor \frac{j-1}{2} \rfloor} (-1)^{i+1} * \binom{j-1-i}{i} * w_0^{j-1-2*i} \tag{28}$$

$$\frac{\sin(j * d * h)}{\sin(d * h)} = \sum^{0 \le i \le \lfloor \frac{j-1}{2} \rfloor} (-1)^{i} * \binom{j-1-i}{i} * w_0^{j-1-2*i} \tag{29}$$

The Fibonacci recurrence relation is a special case of the sine recurrence relation with weights w_L and w_0 of identity.

$$F_{j+2} = F_{j+1} + F_j \tag{30}$$

A Fibonacci number F is the sum of the binomial coefficients only.

$$F_j = \sum^{0 \le i \le \lfloor \frac{j-1}{2} \rfloor} \binom{j-1-i}{i} \tag{31}$$

References

[1] Sine Theorem, Hans-Dieter Reuter, http://www.joinedpolynomials.org/sine.pdf

[2] Joined Polynomials, Hans-Dieter Reuter, http://www.joinedpolynomials.org/jp.pdf

[3] http://gmplib.org, visited 15. October 2010

Listing 1: sine theorem in C with gmp [3]

```
#include <assert.h>
#include <math.h>
#include <stdio.h>
#include <gmp.h>

void gbinom(mpf_t r, unsigned const a, unsigned const b)
{
  unsigned i;
  mpf_set_ui(r, 1);
  for(i = 1; i <= b; ++i)
  {
    mpf_mul_ui(r, r, a-i+1);
    mpf_div_ui(r, r, i);
  }
}

int main(void)
{
  double const c = 3.,h = .2, w0 = 2.-h*h*c*c;
  double const d = acos(w0/2.)/h, r = 1./sin(d*h);
  unsigned i, j;
  double s;
  mpf_t e, b, t;
  FILE * f = fopen("gsine.dat", "w");
  assert(f);
  mpf_set_default_prec(1024);
  mpf_init(b); mpf_init(e); mpf_init(t);
  for(j = 1; j < 500; ++j)
  {
    mpf_set_ui(e, 0);
    for(i = 0; 2*i <= j-1; ++i)
    {
      gbinom(b, j-i-1, i);
      if(i%2) { mpf_neg(b, b); }
      mpf_set_d(t, w0);
      mpf_pow_ui(t, t, j-1-2*i);
      mpf_mul(b, b, t);
      mpf_add(e, e, b);
    }
    s = sin(d*(j)*h)*r;
    fprintf(f, "%f %f %f %.24f\n", j*h, mpf_get_d(e), s, mpf_get_d(e)-s);
  }
  fclose(f);
  mpf_clear(b); mpf_clear(e); mpf_clear(t);
  printf("%f*sin(%f*x)\n", r, d);
  return 0;
}
```